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The Laplace Transform Method in an Algorithm of Solving Differential Equations with Delayed Argument

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The method is used for linear differential equations with delayed argument. There is constructed an algorithm, which is symbolic-numerical. The numerical component concerns a representation of functions, involved into the process by some kind of series.

Key words and phrases: linear differential equations, delayed argument, symbolic-numerical algorithm.

1. Introduction

There is a class of physical problems, which is associated with action of some kind of complementary forces - forces which are involved at various not initial time moments. Such problems frequently lead to the so called differential equations with delayed argument. Different ways of dealing with such equations exist. See for example [1, 2]. We consider linear equations with constant coefficients and right-hand parts of exponential increase.

Applications of the Laplace transform method are well known. It permits to reduce an infinitesimal problem to an algebraic one that may be solved symbolically or symbolic-numerically. Moreover, it gives means to estimate an accuracy of calculations. However there are some facts which prevent using this method in a symbolic way. Some difficulties, for example, are connected with a form of the solution of the Laplace image of the input differential equations, i.e. the exponential polynomials, which appears in the solution of algebraic equation. We suggest the usage of series expansion of some kind for symbolic-numerical solution with a necessary accuracy. It extends the class of equations to be solved by this method.

We restrict ourselves to the consideration of one equation, but the method works similarly with systems of equations of such type.

2. A differential equation with delayed argument and application of Laplace transform

We consider all functions, either unknown or standing at the right-hand parts of equations, on the segment $\mathbf{T} : 0 \leq t \leq T$. Split $\mathbf{T}$ into parts by rational points $0 < t_k < t_{k+1} < T, k = 0, \dots, N$. All functions of the argument t are supposed to satisfy the conditions for existing of their Laplace transform, i.e. they have an exponential increase. Consider an equation

$$x^{(n)}(t) + \sum_{j=1}^{n} \sum_{k=0}^{N} a_{jk} x^{(n-j)}(t - t_k) = f(t), \tag{1}$$

with initial conditions $x^{(n-j)}(0) = x_0^{(n-j)}, j = 1..n$. As the right-hand members of equations we consider here a composite function $f(t)$, whose components are represented as finite sums of exponents with polynomial coefficients. $f(t) = f_k(t),\ t_k < t < t_{k+1},\ k = 1..N$, where $f_k(t) = \sum_{s_k=1}^{S_k} P_{s_k}(t) e^{b_{s_k} t},\ k = 1..N$, and $P_{s_k}(t) = \sum_{m=0}^{M_{s_k}} c_{s_k m} t^m$.

The first step is to prepare the equation (1) for performance of Laplace transform. Using the Heaviside function $\eta(t)$ we obtain the following form of the equation (1)

$$x^{(n)}(t)+\sum_{j=1}^{n}\sum_{k=0}^{N}a_{jk}\eta(t-t_k)x^{(n-j)}(t-t_k)=f(t),$$

$f(t)$ must also be written by means of Heaviside function.

It permits to write symbolically the Laplace image of the equation (1):

$$\left(p^n+\sum_{j=1}^{n}\sum_{k=0}^{N}a_{jk}e^{-pt_k}p^{n-j}\right)X(p)=$$

$$\sum_{j=1}^{n}p^{j-1}x_0^{(n-j)}+\sum_{j=1}^{n-1}\sum_{k=0}^{N}a_{jk}p^{j-1}x_0^{(n-j)}e^{-pt_k}+F(p),$$

where $X(p)$ and $F(p)$ are the Laplace images of $x(t)$ and $f(t)$, correspondingly, and $F(p)$ is also a sum of exponents with polynomial coefficients. Denote

$$Q(p)=\sum_{j=1}^{n}p^{j-1}x_0^{(n-j)}+\sum_{j=1}^{n-1}\sum_{k=0}^{N}a_{jk}p^{j-1}x_0^{(n-j)}e^{-pt_k}+F(p),$$

$$D(p)=p^n+\sum_{j=1}^{n}\sum_{k=0}^{N}a_{jk}e^{-pt_k}p^{n-j},\text{ then }X(p)=\frac{Q(p)}{D(p)}.$$

The last step of the algorithm is the Inverse Laplace transform. We must find a half-plane to chose a vertical line for inverse transform and for a series expansion.

Consider $X(p)$ and its denominator $D(p)$. There exists a half-plane, where $X(p)$ is holomorphic. To find it we must find a half-plane, where $D(p)$ is non-zero. Let us find $\sigma>0$ such that $D(p)\neq 0$ for all $p:\mathrm{Re}\,p>\sigma$. As $D(p)\to\infty$ while $p\to\infty$ then for each $\delta>0$ there exists σ such that $D(p)>\delta$ if $p:\mathrm{Re}\,p>\sigma$.

We have for sufficiently large $|p|$

$$|D(p)|\geq|p|^N(1-\sum_{j=1}^{n}\sum_{k=0}^{N}|a_{jk}||p|^{(n-j)/N}).$$

Denote $A=\sum_{j=1}^{n}\sum_{k=0}^{N}|a_{jk}|$, and take $\sigma=max\left\{\delta,\frac{\delta}{1-A}\right\}$. If $\mathrm{Re}\,p>\sigma$, then $|D(p)|>\delta$. So we may take the half-plane $\mathrm{Re}\,p>\sigma$, $X(p)$ is holomorphic in it.

We must mention, that the line $\mathrm{Re}\,p=\tilde{\sigma},\tilde{\sigma}\geq\sigma$, may be taken as line of integration for numerical calculation of the inverse Laplace transform.

At last we must expand the solution in one special series Writing t_k as $t_k=\frac{\tau_k}{\sigma_k}$, denote $\sigma=LCM_k(\sigma_k)$, and $t_k=\frac{\tilde{\tau}_k}{\sigma}$. Denote $e^{-\frac{p}{\sigma}}=z$. Then

$$X(p)=\frac{\sum_{j=1}^{n}p^{j-1}x_0^{(n-j)}+\sum_{j=1}^{n-1}\sum_{k=0}^{N}a_{jk}p^{j-1}x_0^{(n-j)}z^{\tilde{\tau}_k}+F(p)}{p^n+\sum_{j=1}^{n}\sum_{k=0}^{N}a_{jk}z^{\tilde{\tau}_k}p^{n-j}}.\qquad(2)$$

We do not write the exact expression of such kind for $F(p)$, as it is rather bulky, mention only, that the exponents are the same, because we take the same split points.

Formally we expand (2) in a Taylor series by z at the point $z = 0$. It corresponds to $p : \mathrm{Re}\, p = +\infty$. Substituting $e^{-\frac{p}{\sigma}}$ instead of z, we obtain the series for $X(p)$ by $e^{-\frac{np}{\sigma}}$, which converges in some neighbourhood of ∞:

$$\sum_n A_n e^{-\frac{np}{\sigma}}, \qquad (3)$$

where A_n are proper fractions, and can be represented as sums of partial fractions.

For the series (3) the Inverse Laplace transform may be written symbolically. A problem is to define n and Re p sufficient for designed accuracy of the differential equation.

Let us take the $n - th$ Taylor approximation of $X(p)$ and find its inverse Laplace image. Denote by $\tilde{x}(t)$ an approximate solution of (1), which is equal to this image. The accuracy of such solution we denote by ϵ, i.e. $\max_{\mathbf{T}}|x(t) - \tilde{x}(t)| < \epsilon$.

The remainder term of (3) may be written in the form $\sum_{k=n} \frac{\alpha_k}{p^k} e^{-\frac{kp}{\sigma}}$. Demand $|p||\alpha_n|/(\mathrm{Re}\ p)^n e^{-(n\mathrm{Re}\ p)/\sigma} < \epsilon$. Then we obtain it for each $t \in \mathbf{T}$.

3. Conclusions

In the conclusion let us mention the advantages of our method:

1. The algebraization of the problem makes possible to apply fast and efficient method for solving algebraic linear system with polynomial coefficients. It is actual because it permits to solve huge problems.

2. The expansion into the series of exponent with polynomial coefficients extends the class of equations which may be solved by means of Laplace transform.

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Преобразование Лапласа при решении дифференциальных уравнений с запаздывающими аргументами

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Предлагается символьно-численный алгоритм решения дифференциальных уравнений с запаздывающим аргументом. Численная компонента алгоритма содержит представление функций некоторым специальным рядом.

Ключевые слова: линейные дифференциальные уравнения, запаздывающий аргумент, численно-символьный алгоритм.