To the problem of constructing an algorithm for symbolic integration[1]

Svetlana Tararova

Abstract. This article is devoted the integration of elementary functions. The integral is the sum of a polynomial $p(x)$ and rational function $\frac{q}{r}$. Here we consider only the polynomial part of the integral, in which the polynomial $p(x) = \sum_{j=0}^{m} A_j \theta_n^j$, where θ_n is a logarithmic or exponential function over a differential field $F(x, \theta_1, \ldots, \theta_{n-1})$ and the coefficients A_j $(j = 0, \ldots, m)$ is a logarithmic or exponential function over a differential field $F(x, \theta_1, \ldots, \theta_{n-2})$.

The fundamental result on elementary function integration was first presented by Liouville [1] in 1833 and is a generalization of the above statement. It is the basis of the algorithmic approach to elementary function integration.

Theorem (Liouville's Principle). *Let F be a differential field with constant field K. For $f \in F$ suppose that the equation $g' = f$ (i.e. $g = \int f$) has a solution $g \in G$ where G is an elementary extension of F having the same constant field K. Then there exist $v_0, v_1, \ldots, v_m \in F$ and constants $c_1, c_2, \ldots, c_m \in K$ such that*

$$f = v_0' + \sum_{i=1}^{m} c_i * \frac{v_i'}{v_i}.$$

In other words, such that

$$\int f = v_0 + \sum_{i=1}^{n} c_i \ln(v_i).$$

We formulate the problem of integration in finite terms. Let $F(x, \theta_1, \ldots, \theta_n)$ be a differential field with constant field K where x is an variable such that $x' = 1$ and for any where each $i = 1, \ldots, n$ element θ_i is transcendental and either logarithmic or exponential over the field $F_{n-1} = F(x, \theta_1, \ldots, \theta_{n-1})$.

We construct an algorithm that allows for any elementary function $f \in F_n$ to find the elementary function $g(x)$ for which $g'(x) = f$, or to prove that this

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function does not exist. Integration algorithm is a recursive character, that is, the problem formulated in terms of the field F_i, you need to go to one or more tasks over the field F_{i-1}.

Logarithmic Extension: Integration of the Polynomial Part.

Consider now the case where we are integrating the polynomial part in a logarithmic extension. For an integrand $f \in F(x, \theta_1, \ldots, \theta_n)$ where the last extension $\theta = \theta_n$ is logarithmic over $F(x, \theta_1, \ldots, \theta_{n-1})$ (specifically, $\theta = \ln(u)$ where $u \in F_{n-1}$), the polynomial part is a polynomial $p(\theta) \in F_{n-1}[\theta]$. Let $p(\theta) = \sum_{i=0}^{n} p_i \theta^i$ where $p_i \in F_{n-1}$. By Liouville's Principle, if $\int p(\theta)$ is elementary then

$$p(\theta) = v_0'(\theta) + \sum_{i=1}^{m} c_i \frac{v_i'(\theta)}{v_i(\theta)}.$$

Let $v_0(\theta) = \sum_{i=0}^{n+1} B_i \theta^i$ where $B_i \in F_{n-1}$. Applying the differentiation and equating coefficients of like powers of θ yields the following system of equations:

$$\sum_{i=0}^{n} p_i \theta^i = (\sum_{i=0}^{n+1} B_i \theta^i + \sum_{i=1}^{m} c_i \ln(v_i(\theta)))' = \sum_{i=0}^{n+1} B_i' \theta^i + \sum_{i=1}^{n+1} i B_i \theta' \theta^{i-1} + \sum_{i=1}^{m} c_i \frac{v_i'(\theta)}{v_i(\theta)}.$$

Using the method of undetermined coefficients we obtain B_i $(i = 1, \ldots, n+1)$.

Exponential Extension: Integration of the Polynomial Part.

The "polynomial part" of an integrand $f \in F(x, \theta_1, \ldots, \theta_n)$, in the case where the last extension $\theta = \theta_n$ is exponential over $F(x, \theta_1, \ldots, \theta_{n-1})$, is an "extended polynomial"

$$p(\theta) = \sum_{i=-k}^{l} p_i \theta^i,$$

where $p_i \in F_{n-1}$ which may contain both positive and negative powers of θ. We now consider the problem of computing $\int p(\theta)$ where $\theta = \exp(u)$ for some $u \in F_{n-1}$. By Liouville's Principle, if $p(\theta)$ has an elementary integral then

$$p(\theta) = v_0'(\theta) + \sum_{i=1}^{m} c_i \frac{v_i'(\theta)}{v_i(\theta)}.$$

Let $v_0(\theta) = \sum_{i=-k}^{l} q_i \theta^i$ where $q_i \in F_{n-1}$. The process of integration is to find the coefficients q_i $i = -k, \ldots, l$ of the equation:

$$\sum_{i=-k}^{l} p_i \theta^i = (\sum_{i=-k}^{l} q_i \theta^i + \sum_{i=1}^{m} c_i \ln(v_i(\theta)))' = \sum_{i=-k}^{l} q_i' \theta^i + \sum_{i=-k}^{l} i B_i \theta^i + \sum_{i=1}^{m} c_i \frac{v_i'(\theta)}{v_i(\theta)}.$$

Applying the equating coefficients of like powers of θ yields the following system of equations:

$$p_i = q_i' + i * u' * q_i \text{ for } j \neq 0, \tag{1}$$

$$p_0 = q_0 + \sum_{i=1}^{m} c_i \frac{v_i'(\theta)}{v_i(\theta)} = \bar{q}.$$

To solve the data conn differential equation is first necessary to determine the type q_i, where q_i is a rational function. Let $q_i = \frac{a}{b}$ where a and $b \in F_{n-1}$. If q_i had a non-constant denominator, the degree of that denominator would increase in q_i' and could not be canceled by the other terms.
The following cases

1. p_i is a polynomial, $p_i \in F(x, \theta_1, \ldots, \theta_{n-1})$, then q_i will also be a polynomial $q_i = a$, where $a \in F(x, \theta_1, \ldots, \theta_{n-1})$.
   - if u' is also a polynomial and $u' = v$, where $v \in F(x, \theta_1, \ldots, \theta_{n-1})$, the equation (1) takes the following form:
   $$p_i = a' + iva. \tag{2}$$
   - if u' is a rational function $u' = \frac{v}{w}$, where $v,\ w \in F(x, \theta_1, \ldots, \theta_{n-1})$ and $\deg(v) < \deg(w))$, the equation (1) takes the form:
   $$p_i w = wa' + iva. \tag{3}$$
2. p_i is a rational function, i.e. $p_i = \frac{r}{g}$, where $r, g \in F_{n-1}$, then q_i will also be a rational function $q_i = \frac{a}{b}$, where $a, b \in F(x, \theta_1, \ldots, \theta_{n-1})$ and $\deg(a) < \deg(b)$.
   - if u' is a polynomial $u' = v$, where $v \in F(x, \theta_1, \ldots, \theta_{n-1})$, the equation (1) takes the following form:
   $$\frac{r}{g} = \frac{a'b - ab'}{b^2} + \frac{iva}{b}.$$
   Then
   $$b = \sqrt{g}, r = a'b - ab' + ivab = ba' + (-b' + ivb)a. \tag{4}$$
   - if u' is a rational function $u' = \frac{v}{w}$, where $v, w \in F(x, \theta_1, \ldots, \theta_{n-1})$ and $\deg(v) < \deg(w)$, the equation (1) takes the form:
   $$\frac{r}{g} = \frac{a'b - ab'}{b^2} + \frac{iva}{wb}$$
   and
   $$b = \sqrt{\frac{g}{w}}, r = w(a'b - ab') + ivab = wba' + (-wb' + ivb)a. \tag{5}$$

Denote in equations (2), (3), (4), (5) the left side through the P, and by $\bar{c}_1$ the coefficient of a', and the coefficient of a by $\bar{c}_2$. Finally, we obtain the equation:
$$P = \bar{c}_1 a' + \bar{c}_2 a, \tag{6}$$
where $P, \bar{c}_1, \bar{c}_2, a \in F(x, \theta_1, \ldots, \theta_{n-1})$.

We find a polynomial a. First, will assess the degree of the polynomial a. Suppose $\deg(a) = h$, then, if θ_{n-1} is either logarithmic or exponential extension then $\deg(a') = h$. In other cases $\deg(a') = h - 1$.

Consider three cases.

1) If $P, \bar{c}_1, \bar{c}_2, a \in F(x)$, then $h = \deg(P) - \max(\max(0, \deg(\bar{c}_1) - 1), \deg(\bar{c}_2))$. If $h < 0$, then starting integrand is not integrable in elementary form. If $h = 0$, then $a = \frac{P}{\bar{c}_2}$. If $h > 0$, then $a = \sum_{i=0}^{h} a_i x^i$. Then the equation (6) takes the form:

$$P = \bar{c}_1(a_0 + \sum_{i=1}^{h} i a_i x^{i-1}) + \bar{c}_2 \sum_{i=0}^{h} a_i x^i.$$

Using the method of undetermined coefficients can be found a_i $i = 0, \ldots, h$.

2) If θ_{n-1} is a logarithmic extension, then $h = \deg(P) - \max(\deg(\bar{c}_1), \deg(\bar{c}_2))$. If $h < 0$, then starting integrand is not integrable in elementary form. If $h \geqslant 0$, then $a = \sum_{i=0}^{h} a_i \theta_{n-1}^i$. Then the equation (6) takes the form:

$$P = \sum_{i=0}^{t} \bar{p}_i \theta_{n-1}^i = \bar{c}_1(\sum_{i=0}^{h} a_i' \theta_{n-1}^i + \sum_{i=1}^{h} i a_i \theta_{n-1}' \theta_{n-1}^{i-1}) + \bar{c}_2 \sum_{i=0}^{h} a_i \theta_{n-1}^i.$$

Equating the left and right sides of this equality, we obtain a system of differential equations for the a_i $i = 0, \ldots, h$, where $a_i \in F(x, \theta_1, \ldots, \theta_{n-2})$.

3) If θ_{n-1} is a exponential extension, then in addition to the highest degree of the polynomial a and find the lowest degree of the polynomial a, as $\exp^{-1}(x)$ is also related to the polynomial part. Let $\bar{c}_1 = \sum_{i=0}^{k1} \bar{c1}_i \theta_{n-1}^i$ and $\bar{c}_2 = \sum_{i=t2}^{t1} \bar{c2}_i \theta_{n-1}^i$, then equation (6) takes the form:

$$P = \sum_{i=m2}^{m1} \bar{p}_i \theta_{n-1}^i = \sum_{i=0}^{k1} \bar{c1}_i \theta_{n-1}^i \sum_{i=h2}^{h1} (a_i' + i a_i \mu') \theta_{n-1}^i + \sum_{i=t2}^{t1} \bar{c2}_i \theta_{n-1}^i \sum_{i=h2}^{h1} a_i \theta_{n-1}^i,$$

where μ is a argument of the function θ_{n-1}. Then $h1 = m1 - \max(k1, t1)$ and $h2 = m2 - \max(0, t2)$. It follows that $a = \sum_{i=h2}^{h1} a_i \theta_{n-1}^i$.

Equating the left and right sides of this equality, we obtain a system of differential equations for the a_i $i = h2, \ldots, h1$, where $a_i \in F(x, \theta_1, \ldots, \theta_{n-2})$.

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Svetlana Tararova
e-mail: tararovasveta@gmail.com