One approach to symbolic solution of partial differential equations[1]

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Abstract. The work is devoted to solving partial differential equations with constant coefficients which admit a Laplace-Carson transform for functions on the right part.

1. Introduction

The work is devoted to solving partial differential equations with constant coefficients that admit a Laplace-Carson transform for function $h(x)$ on the right part. This is a function of several variables $h(x),\ x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. This function is bounded, having a finite number of discontinuity points of type I and no more than exponential growth rate of each variable on ${\mathbb{R}_+}^n$ and equals 0 at all other points on $\mathbb{R}^n$. The class of such functions is denoted by $\mathbf{S_n}$.

The solution is based on the Laplace-Carson transform. Algorithm consists of two stages. The first stage, specifies requirements for the initial conditions. The second stage, calculation of the initial conditions, the substitution of images depicting the initial conditions in equation, the original finding of the desired function from the inverse Laplace-Carson transform.

2. Determine the initial conditions

Consider the differential equation in partial derivatives on $\mathbb{R}^2_+$:

$$\sum_{n=0}^{m} a_n \frac{\partial^m}{\partial x^n \partial y^{m-n}} f(x,y) = h(x,y), a_n \in \mathbb{R},\ h(x,y) \in \mathbf{S_n},\ m \in \mathbb{N}. \tag{1}$$

Laplace-Carson transform is given by:

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$$LC: f(x,y) \mapsto u(p,q) = pq \int\limits_0^\infty \int\limits_0^\infty e^{-px} e^{-qy} f(x,y)dxdy,$$

$p = \sigma + i\mu, q = \tau + iv$ — complex parameters.

We know [1]:

$LC: \dfrac{\partial^n}{\partial x^n} f(x,y) \mapsto p^n u(p,q) - \sum\limits_{k=0}^{n-1} p^{n-k} u_{2,x^k}(0,q),$

$LC: \dfrac{\partial^n}{\partial y^n} f(x,y) \mapsto q^n u(p,q) - \sum\limits_{k=0}^{n-1} q^{n-k} u_{1,y^k}(p,0),$

$LC: \dfrac{\partial^{m+n}}{\partial x^m \partial y^n} f(x,y) \mapsto p^m q^n u(p,q) - p^m \sum\limits_{l=0}^{n-1} q^{n-l} u_{1,y^l}(p,0) - q^n \sum\limits_{k=0}^{m-1} p^{m-k} u_{2,x^k}(0,q) +$
$+ \sum\limits_{k=0}^{m-1} \sum\limits_{l=0}^{n-1} p^{m-k} q^{n-l} f_{x^k y^l}^{(k+l)}(0,0),$

$m, n \geq 1; u_{2,x^k}(0,q) = q \int\limits_0^\infty e^{-q\eta} \left. \dfrac{\partial^k f(x,\eta)}{\partial x^k} \right|_{x=0} d\eta; u_{1,y^k}(p,0) = p \int\limits_0^\infty e^{-q\xi} \left. \dfrac{\partial^k f(\xi,y)}{\partial y^k} \right|_{y=0} d\xi.$

After the Laplace-Carson's left-hand side of equation (1), we need to know the following functions, which are called initial conditions and can be given in the form:

$$\left. \frac{\partial^k f(x,y)}{\partial x^k} \right|_{x=0} = a_k(y), \quad \left. \frac{\partial^h f(x,y)}{\partial y^h} \right|_{y=0} = b_h(x), \tag{2}$$

$k = 0, \ldots, n-1$; n — order of the derivative of f in the variable x in (1); $h = 0, \ldots, m-1$, m — order of the derivative of f in the variable y in (1).

3. Definition of a family of initial conditions

Let the unknown function $f = f(x,y)$ in equation (1), k — highest derivative of f in the variable x, m — highest derivative of f in the variable y.

In accordance with the (2):

$$IC = \{a_0, \ldots, a_{k-1}, b_0, \ldots, b_{m-1}\},$$

$a_i = \left. \dfrac{\partial^i f(x,y)}{\partial x^i} \right|_{x=0}$, $b_j = \left. \dfrac{\partial^j f(x,y)}{\partial y^j} \right|_{y=0}$; $i = 0, \ldots, k-1$; $j = 0, \ldots, m-1, a_i, b_j \in \mathbf{S_2}$.

After the Laplace-Carson transform:

$$LC: f(x,y) \mapsto u(p,q); \ a_i(y) \mapsto \alpha_i(q); \ b_j(x) \mapsto \beta_j(p).$$

After the Laplace-Carson transform, and the resulting solution of the algebraic equation we get an image of the required u of f as a rational function:

$$u = \frac{\Omega(p,q)}{Q(p,q)}, \tag{3}$$

$\Omega(p,q) = \sum\limits_{i=0}^{k-1} P_i(p,q)\alpha_i + \sum\limits_{j=0}^{m-1} P_j(p,q)\beta_j + \Theta(p,q).$

Let the denominator $Q(p,q)$ can be written as $Q(p,q) = \prod\limits_{r=1}^{l} (p-\psi_r(q))$, $l \in \mathbb{N}$, and, $\psi_i(q) \neq \psi_j(q)$, where $i \neq j$, $i,j = 1,\ldots,l$.

Inverse Laplace-Carson transform is given by:

$$LC^{-1} : u(p,q) \mapsto f(x,y) = \int\limits_{\sigma-i\infty}^{\sigma+i\infty} \int\limits_{\tau-i\infty}^{\tau+i\infty} \frac{u(p,q)}{pq} e^{px+qy} dpdq,$$

$p = \sigma + i\mu, q = \tau + iv$ — complex parameters.

To calculate the original f by applying the inverse Laplace-Carson transform requires that $\exists m,\ n \in \mathbb{R}^2$ such that $|u| < \infty$ for all $Re(p) > n > 0$, $Re(q) > m > 0$. The expression $p - \psi_r(q)$ turns to zero on $Re(p) > n > 0$, $Re(q) > m > 0$, $m,\ n \in \mathbb{R}^2$, if and only if, when the function $\psi_r(q)$ satisfies the following conditions

$$\begin{cases} \lim\limits_{\rho->+\infty} |\psi_r(q)| = +\infty, \\ \Pi/2 \leq Arg(\lim\limits_{\rho->+\infty} (\psi_r(q), q)) \leq 0, \end{cases} \tag{4}$$

$q = \rho e^{i\varphi}$, $0 < \varphi < \Pi/2$, $r = 0,\ldots,l$.

Therefore, such a function $\psi_r(q)$, which satisfy the system (4) should be included as factors in the decomposition of the numerator of (3).

Let us write $Q(p,q)$ in the form $Q(p,q) = Q_1(p,q) * Q_2(p,q)$, where $Q_1(p,q) = \prod\limits_{c=1}^{d} (p - \psi_c(q))$, $Q_2(p,q) = \prod\limits_{e=d+1}^{l} (p - \psi_e(q))$, $\psi_c(q)$ satisfy (4), and $\psi_e(q)$ does not satisfy (4). We substitute the functions $\psi_c(q)$ in $\Omega(p,q)$. We obtain

$$\begin{cases} \Omega(\psi_1(q), q) = 0, \\ \ldots \\ \Omega(\psi_d(q), q) = 0. \end{cases} \tag{5}$$

We remark that α_i and β_j are Ω is linear, so (5) is an inhomogeneous system of linear equations for the functions α_i and β_j.

Let T matrix of coefficients of the main system (5), $rank(T) = t$, where $t \leq d \leq k + m - 2$. Then we can express the system (5) t unknowns as linear combinations of the remaining $k + m - 2 - t$ free unknowns, and the coefficients are rational functions.

As the free variables can serve any function in $\mathbf{S_n}$. These free go into the function of the desired solution of the differential equation (1) in general form.

References

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