Computing general solutions of systems of ordinary differential equations with constant coefficients[1]

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Abstract. We discuss the algorithm for obtain the symbolic analytical solution of inhomogeneous system of ordinary linear differential equations with constant coefficients. The developed algorithm may be used for obtaining partial and general solutions of differential equations in an analytical form. It may be used for obtaining the solution with the required accuracy. This algorithm is efficient for the solution of large systems of differential equations. This algorithm is a part of the Mathpar computer algebra system. We demonstrate several examples in the Mathpar system.

1. Production of the problem

Given a heterogeneous system of ordinary linear differential equations with constant coefficients:

$$\sum_{j=1}^{n} D_{ji}(t)x_i(t) = f_i(t), D_{ji}(t) = \sum_{k=0}^{m} a_{kji}\frac{d^k}{dt^k}, i = 1, ..., m, a_{kji} \in \mathbb{R}, n, m \in \mathbb{N}, \quad (1)$$

where a_{kji} — real numbers, $f_i(t), x_i(t)$ — bounded on $\mathbb{R}_+$ function, having a finite number of discontinuities I race and satisfying the conditions: $f_i(t) \equiv 0$ at $t < 0$, $|f_i(t)| < Me^{s_0 t}$ at $t > 0$, where $M > 0$, $s_0 \geq 0$ — some real constants.

The system (1) can be written in matrix form:

$$A(t)X(t) = F(t), \quad (2)$$

where

$$A(t) = \begin{pmatrix} D_{11}(t) & D_{12}(t) & \dots & D_{1m}(t) \\ D_{21}(t) & D_{22}(t) & \dots & D_{2m}(t) \\ \dots & \dots & \dots & \dots \\ D_{n1}(t) & D_{n2}(t) & \dots & D_{nm}(t) \end{pmatrix},$$

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$$X(t) = [x_1(t), ..., x_n(t)]^T, \; F(t) = [f_1(t), ..., f_n(t)]^T.$$

Let $x_i^{(k)}(t)$ denotes the k derivative of the function $x_i(t)$, a number of x_{0i}^k define the initial conditions:

$$x_i^{(k)}(0) = x_{0i}^k, \tag{3}$$

where $k = 1, 2, ..., m-1, \; i = 1, 2, ..., m, \; x_{0i}^k \in \mathbb{R}$.

We assume, that in general each of the functions $f_i(t)$ on the right side may take the form of a finite sum:

$$f_i(t) = \sum_j p_{ij}(t) e^{\delta_{ij} t} \sin^{\mu_{ij}}(\gamma_{ij} t) \cos^{\nu_{ij}}(\beta_{ij} t) UnitStep(t - \alpha_{ij}),$$

where $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij} \in \mathbb{R}, \mu_{ij}, \nu_{ij} \in \mathbb{N}, p_{ij}(t)$ — polynomial variable t, function $UnitStep(t)$ takes the value 1 for a non-negative argument and a value of 0 for the remaining.

Required to find a solution X(t) systems (1) or (2), satisfying the conditions (3), in an analytical form.

The solution of system (1) called private, when the initial conditions are given numbers. The solution of system (1) called the general solution, if the initial conditions included in the decision to free variables, which can be chosen so, the solution $X(t)$ satisfy arbitrary initial conditions.

2. Algorithm of the Laplace

The algorithm consists of three stages [1-8].

Step I. The direct Laplace transform of differential equations.

The Laplace transform of the function $f(t)$:

$$\mathcal{F}(p) = \int_0^\infty f(t) e^{-pt} dt, \quad p \in \mathbb{C}. \tag{4}$$

As a result of the transformation functions, standing on the left and right parts of the system of differential equations (1), and initial conditions (3) by the formula (4) obtain a system of algebraic equations:

$$\mathcal{A}(p)\mathcal{X}(p) - \mathcal{B}(p) = \mathcal{F}(p). \tag{5}$$

Here $\mathcal{A}(p)$ — image of the left side of the system (1), $\mathcal{F}(p)$ — image of the right-hand side of (1), $\mathcal{B}(p)$ — vector, appearing with the introduction of initial conditions (3).

Step II. The solution of the algebraic system.

The solution of the algebraic system (5) sought in the form

$$\mathcal{X}(p) = \mathcal{A}(p)^{-1}(\mathcal{F}(p) + \mathcal{B}(p)). \tag{6}$$

The result is a vector of of rational functions of p.

Step III. The inverse Laplace transform.

The inverse Laplace transform of the function $\mathcal{F}(p)$:

$$f(t) = L^{-1}\{\mathcal{F}\} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{pt}\mathcal{F}(p)dp.$$

The desired solution (1) given by the vector $X(t)$:

$$X(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{pt}\mathcal{X}(p)dp. \tag{7}$$

3. Direct Laplace transform

Let $\mathcal{F}(p)$ — image of the function $f(t)$ at direct Laplace transform. As a result, the direct conversion Laplace derivative n order of the function $f(t)$ we get polynomial of degree n

$$\int_0^\infty f^{(n)}(t)e^{-pt}dt = p^n\mathcal{F}(p) - p^{n-1}f(0) - p^{n-2}f^{'}(0) - \ldots - f^{(n-1)}(0). \tag{8}$$

As a result, the direct left of the Laplace transform part of a system of differential equations is transformed into a matrix polynomials $\mathcal{A}(p)$ one real variable p.

$$\int_0^\infty \sum_{j=1}^{n}\sum_{k=0}^{m} a_{kji}\frac{d^k}{dt^k}x_i(t)e^{-pt}dt = \sum_{j=1}^{n}\mathcal{D}_{ji}(p)x_i(p),$$

where $\mathcal{D}_{ji}(p) = \sum_{i=1}^{m} a_{kji}p^k$.

$$\mathcal{A}(p) = \begin{pmatrix} \mathcal{D}_{11}(p) & \mathcal{D}_{12}(p) & \ldots & \mathcal{D}_{1m}(p) \\ \mathcal{D}_{21}(p) & \mathcal{D}_{22}(p) & \ldots & \mathcal{D}_{2m}(p) \\ \ldots & \ldots & \ldots & \ldots \\ \mathcal{D}_{n1}(p) & \mathcal{D}_{n2}(p) & \ldots & \mathcal{D}_{nm}(p) \end{pmatrix}. \tag{9}$$

Direct result of the Laplace transform (4) right-hand side is the vector functions.

$$\mathcal{F}(p) = [f_1(p), \ldots, f_n(p)]^T,$$

Each of the functions $f_i(p)$ is the sum of products exponential and rational functions

$$f_i(p) = \sum_{j=1}^{n}\frac{Q_{ij}(p)e^{\alpha_{ij}p}}{L(p)}, \text{ where } Q_{ij}(p), L(p) \text{ — polynomials.} \tag{10}$$

The result of the transformation of the initial conditions is the sum of products of polynomials p on the free variables, denoting the initial conditions (3):

$$\mathcal{B}(p) = \sum_{i=1}^{n}\sum_{k=0}^{m-1} d_{ki}(p)x_{0i}^{k}, \quad d_{ik}(p) = \sum_{i=k}^{m-1} a_{i+1,i}p^{i-k}. \tag{11}$$

As a result the Laplace transform of the system of ordinary linear differential equations, we obtain algebraic system of linear equations

$$\mathcal{X}(p) = \mathcal{A}(p)^{-1}(\mathcal{F}(p) + \mathcal{B}(p)).$$

4. Solution of the algebraic system of linear equations

For a matrix of polynomials $\mathcal{A}(p) \in \mathbb{C}[p]^{n\times m}$, given by the formula (9), compute the inverse matrix $\mathcal{A}^{-1} = \mathcal{A}^*(p)/det(\mathcal{A}(p))$, where $\mathcal{A}^*(p)$ — adjoint matrix, $det(\mathcal{A}(p))$ — determinant.

Polynomial $det(\mathcal{A}(p))$ decompose into linear factors in of $\mathbb{C}$:

$$det(\mathcal{A}(p)) = \prod_{k=0}^{r} (p - p_k)^{\epsilon_k}, \text{ where } r \leq nm,\ p_k \in \mathbb{C},\ \epsilon_k \in \mathbb{N}. \tag{12}$$

Spread the fraction $1/det(\mathcal{A}(p))$ in the sum of simple fractions in of $\mathbb{C}$:

$$\frac{1}{det(\mathcal{A}(p))} = \sum_{k=0}^{r} \frac{\zeta_k}{(p-p_k)^{\epsilon_k}}, \text{ where } \zeta_k \in \mathbb{C}. \tag{13}$$

We denote $H(p) = \mathcal{F}(p) + \mathcal{B}(p)$ and $H(p) = [h_1, \ldots, h_n]^T$. then, using the formula (10) and (11), we get:

$$h_i = \sum_{j=1}^{n} \frac{Q_{ij}(p)e^{\alpha_{ij}p}}{L(p)} + \sum_{i=1}^{n}\sum_{k=0}^{m-1} d_{ki}(p)x_{0i}^k, d_{ik}(p) = \sum_{i=k}^{m-1} a_{i+1,i}p^{i-k}.$$

Let $V = [v_1, \ldots, v_n]^T$ and V $= \mathcal{A}^*(p) \cdot H$. Then:

$$v_i = \sum_{j=1}^{n} \frac{U_{ij}(p)e^{\alpha_{ij}p}}{O(p)}, \text{ where } U_{ij}(p),\ O(p) \text{ — polynomials.}$$

We expand v_i the sum of simple fractions in $\mathbb{C}$:

$$v_i = \sum_{k=0}^{r} \frac{\xi_k e^{\alpha_{ij}p}}{(p-p_k)^{\tau_k}}, \text{ where } \xi_k \in \mathbb{C},\ \tau_k \in \mathbb{N}. \tag{14}$$

Let us find a solution $\mathcal{X}(p) = V \cdot 1/det(\mathcal{A}(p))$,

$$\mathcal{X}(p) = V \cdot \sum_{k=0}^{r} \frac{\zeta_k}{(p-p_k)^{\epsilon_k}}.$$

Let $\mathcal{X}(p) = [\chi_1, \ldots, \chi_n]^T$. Then

$$\chi_i(p) = \sum_{k=0}^{r} \frac{\eta_k e^{\alpha_{ij}p}}{(p-p_k)^{\epsilon_k}}, \text{ where } \eta_k \in \mathbb{C}. \tag{15}$$

Remark 1. Expansion (12) factorization of the determinant $det(\mathcal{A}(p))$ requires the calculation of the roots of a polynomial. The roots are approximately. The accuracy of the calculations roots affects the error of the solution $X(t)$ systems (1). The dependence of the error of the obtained solution systems (1) on the accuracy of the roots investigated in [8].

5. Inverse Laplace transform

For each element of the vector $\mathcal{X}(p) = [\chi_1(p), \ldots, \chi_n(p)]^T$ find the inverse transformation Laplace (7) $X(t) = [x_1(t), \ldots, x_n(t)]^T$:

$$x_i(t) = L^{-1}\{\chi_i(p)\}, \text{ where } i = 1, 2, \ldots, n.$$

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