An algorithm for symbolic solving of partial differential equations.

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Abstract. There is produced an algorithm for symbolic solving of partial differential equations with constant coefficients of any order by means of multivariate Laplace-Carson transform. The algorithm is suitable whether for a single equation or for a system of equations. Second order equations of various types may be solved. Boundary conditions are considered according to compatibility conditions.

Consider the space S of functions $f(x), x = (x_1, \dots x_n) \in \mathbf{R}^n_+$, $\mathbf{R}^n_+ = \{x : x_i \geq 0, i = 1, \dots n\}$, for which $M, a = (a_1, \dots, a_n) \in \mathbf{R}^n, a_i > 0, i = 1, \dots n$, exist such that for all $X \in \mathbf{R}^n_+$ the following is true: $|f(x)| \leq Me^{ax}$, $ax = \sum_{i=1}^n a_i x_i$.

On the space S the Laplace-Carson transform (LC) is defined as follows:

$$LC : f(x) \mapsto F(p) = p^1 \int_0^\infty e^{-px} f(x) dx,$$

$p = (p_1, \dots, p_n)$, $p^1 = p_1 \dots p_n$, $px = \sum_{i=1}^n p_i x_i, dx = dx_1 \dots dx_n$.

Consider a system

$$\sum_{k=1}^{K} \sum_{m=0}^{M} \sum_{m_i=0,(m_1+\ldots+m_n=m)}^{m} a^j_{mk} \frac{\partial^m}{\partial^{m_1} x_1 \dots \partial^{m_n} x_n} u_k(x) = f_j(x), \quad j = 1, \dots K, \tag{1}$$

where $u_k(x)$, $k = 1, \dots, K$, – are unknown functions of $x = (x_1, \dots x_n) \in \mathbf{R}^n_+$, $f_j \in S$, a^j_{mk} – numbers.

Example

$$\begin{cases} \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} & = & x, \\ \frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} & = & y, \end{cases}$$

This work was completed with the support of the Sci. Program "Devel. Sci. Potent. High. School", RNP 2.1.1/1853.

$f = f(x, y); \quad g = g(x, y)$.

Initial conditions:

$f(0, y) = a(y); \quad f(x, 0) = b(x); \quad g(0, y) = c(y); \quad g(x, 0) = d(x)$.

$LC : f(x, y) \mapsto u(p, q), \quad g(x, y) \mapsto v(p, q),$
$a(y) \mapsto \alpha(q), \quad b(x) \mapsto \beta(p),$
$c(y) \mapsto \delta(q), \quad d(x) \mapsto \gamma(p).$

$$\begin{array}{rcl} pu - p\alpha(q) + qv - q\gamma(p) & = & 1/p, \\ qu - q\beta(p) + pv - p\delta(q) & = & 1/q. \end{array}$$

Then
$u = -\frac{-\alpha p^2 + \beta q^2 + (\delta - \gamma)pq}{p^2 - q^2}$
$v = -\frac{-p^2 + q^2 + (\alpha - \beta)p^2 q^2 - (\delta p^2 - \gamma q^2)pq}{pq(p^2 - q^2)}$.
The denominator D:
$D(p, q) = pq(p^2 - q^2)$:
The set of zeros of D with infinite limit points at the right half-plane is
$q = p$.

Substituting $q = p$ into the nominators of u and v we obtain the compatibility condition:

$$\alpha - \beta + \gamma - \delta = 0.$$

For example we may take
$\beta = 0; \quad \gamma = \frac{2}{p}; \quad \delta = \frac{2}{q}; \quad \alpha = 0.$
Then

$$u = -\frac{2}{p + q},$$

$$v = -\frac{p + 2p^2 + q + 2q^2 + 2pq}{pq(p + q)}.$$

LC^{-1} :

$$f = -\begin{cases} 2y & , \quad y < x, \\ 2x & , \quad y \geq x, \end{cases}$$

$$g = \begin{cases} (2 + y)x & , \quad y < x, \\ y(2 + x) & , \quad y \geq x. \end{cases}$$

We solve a problem with boundary conditions for each variable. Compatibility conditions may be established by means of Laplace-Carson transform for various types of equations. Some difficulties occur when solving elliptic equations under Dirichlet or Neumann boundary conditions.

References

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