ISSN 1810-0198. Вестник ТГУ, т. 21, вып. 6, 2016

UDC 512.71
DOI: 10.20310/1810-0198-2016-21-6-1933-1937

ON THE COMPUTATION OF LAGRANGE'S QUADRATIC BOUND FOR POSITIVE ROOTS OF POLYNOMIALS

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Lagrange's quadratic bound, LQ, on the values of the positive roots of polynomials consists of two parts. Sorting of one-dimensional arrays is used in the second part of the Lagrange algorithm in all known implementations. We propose to change this part of the algorithm and to avoid the sort. The computing time of the second part as is currently implemented is $O(n \cdot log(n))$. With our improvement we reduce the computing time of the second step of LQ to $O(n)$.
Key words: Lagrange's Quadratic Bound; Positive Roots of Polynomials

Introduction.

On p. 553 of his original paper [1]— or on p. 32, of his famous book [2], which constitutes the 8-th volume of *Œuvres de Lagrange*, edited by Joseph Alfred Serret [3] — Lagrange only states that given the polynomial F , where

$$-\mu y^{r-m} - \nu y^{r-n} - \bar{\omega} y^{r-p} - \cdots$$

are its "negative terms", then an upper bound for the real roots of F is given by the sum of the first two largest of the quantities

$$\sqrt[m]{\mu},\ \sqrt[n]{\nu},\ \sqrt[p]{\bar{\omega}}, \ldots$$

or "a number larger than this sum."

The interesting history of this theorem can be found elsewhere [4]. Here we simply state the theorem without its proof.

T H E O R E M (Lagrange, 1767)

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0,$, be a non constant monic polynomial of degree n over $\mathbb{R}$ and let $a_{n-j} : j \in J$ be the set of its negative coefficients. Then an upper bound for the positive real roots of f is given by the sum of the largest and the second largest members in the set $\left\{ \sqrt[j]{|a_{n-j}|} : j \in J \right\}$. That is,

$$b = \max_{\{a_{n-i}, a_{n-k} \in J\}} \left(\sqrt[i]{|-a_{n-i}|} + \sqrt[k]{|-a_{n-k}|} \right). \qquad (1)$$

Lagrange's Quadratic Algorithm LQ.

In LQ we use the list (of lists) t of length $n+1$, in which initially each entry is $[\,]$, the empty list. The list $t_{n-j} = t[n-j]$ corresponds to the coefficient a_{n-j} of the polynomial and, if $a_{n-j} > 0$, then the list $t[n-j]$ contains all the minimum values produced by $a_{n-j} > 0$ when "paired" with various negative coefficients a_{n-i}, with $i > j$.

An algorithmic description of Lagrange's quadratic method LQ is presented in Algorithm below.

Algorithm 1: `LQ(f, x)`, Lagrange's Quadratic Algorithm.

Input: A univariate polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$, with $a_n > 0$.
Output: An upper bound on the values of the positive roots of $f(x)$.

```
// at least one sign variation ( v ≥ 1 )?
cl ← [ a_0, a_1, a_2, ..., a_{n-1}, a_n ];                 /* list of length n+1 */
v ← number of sign variations in cl ;
if v = 0 then return 0;

// initialize variables
m ← length(cl) ;
t ← [ [ ], [ ], [ ], ..., [ ], [ ] ] ;                      /* list of length n+1 */

// (1) main loop, the same loop in the improved algorithm 2
for j = 0 to m − 1 step 1 do
    if cl(j) < 0 then
        b ← +∞ ;
        index ← m ;
        for k = j + 1 to m − 1 step 1 do
            if cl(k) > 0 then
                q ← (− cl[j]/cl[k])^{1/(k−j)} ;
                if q < b then
                    b ← q ;
                    index ← k ;
                end
            end
        end
        t[index] ← append(t[index], b) ;
    end
end

// (2) secondary loop to process the list of lists t
b ← 0 ;
for j = 0 to m − 1 step 1 do
    tp ← t[j] ;
    if tp ≠ [] then
        if length(tp) = 1 then
            tp ← tp[0] ;                            /* enumeration starts from 0 */
        else
            tp ← sort(tp) ;                         /* sort tp in increasing order */
            tp ← sum(tp[−2 :]) ;                    /* sum of the largest two values */
        end
        if tp > b then
            b ← tp ;
        end
    end
end
return b
```

Algorithm 2: `LQ(f, x)`, Lagrange's Quadratic Algorithm **Improved**.

Input: A univariate polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$, with $a_n > 0$.
Output: An upper bound on the values of the positive roots of $f(x)$.

```
// at least one sign variation ( v ≥ 1 )?
cl ← [ a_0, a_1, a_2, ..., a_{n-1}, a_n ];                /* list of length n+1 */
v ← number of sign variations in cl ;
if v = 0 then return 0;
// initialize variables
m ← length(cl) ;
t ← [ [ ], [ ], [ ], ..., [ ], [ ] ] ;                    /* list of length n+1 */
// (1) main loop, the same loop in the prevoiuse algorithm 1
// (2) secondary loop to process the list of lists t without sort
b ← 0 ;
for j = 0 to m − 1 step 1 do
    if tp ≠ [ ] then
        tp ← t[j] ;
        ltp ← length(tp) ;
        if ltp = 1 then
            sc ← tp[0] ;                      /* enumeration starts from 0 */
        end
        c ← [ ] ;
        if ltp > 2 then
            if tp[0] < tp[1] then
                c ← c[tp[0], tp[1]] ;
            else
                c ← c[tp[1], tp[0]] ;
            end
        else
            c ← c[tp[0], tp[1]] ;
        end
        for k = 2 to ltp − 1 step 1 do
            if tp[k] > c[0] then
                if tp[k] > c[1] then
                    c[0] ← c[1] ;
                    c[1] ← tp[k] ;
                else
                    c[0] ← tp[k] ;
                end
            end
            k ← k + 1 ;
        end
        sc ← sum(c[−2 :]) ;                   /* sum the two values of c */
        if sc > b then
            b ← sc ;
        end
    end
end
return b
```

The algorithm works as follows:

- *each* negative coefficient a_{n-i} of the polynomial is "paired" with *each one* the preceding positive coefficients a_{n-j}, $(i > j)$ and the minimum is taken of all the radicals of the form

$$\sqrt[i-j]{-a_{n-i}/a_{n-j}} \tag{2}$$

  as indicated in Lagrange's theorem); each minimum is then appended to the corresponding list $t[n-j]$,

- we initialize a temporary bound to 0, and then for each non-empty list $t[n-j]$ we proceed as follows: (a) if the list $t[n-j]$ has a single element, and its value is greater than the temporary bound, then *it* (the single element) becomes the temporary bound and, (b) if the list $t[n-j]$ has more than one element, we sort them in increasing order and take the sum of the largest two; if the sum is greater than the temporary bound, then *it* (the sum) becomes the temporary bound; at the end the temporary bound is taken as the estimate of the bound.

Note the function `sort` in step 7 of the Algorithm 1 above. Obviously, the computing time of the secondary loop (steps 5-8) is $O(n \cdot log(n))$.

Improved Lagrange's Quadratic Algorithm LQ.

In the improved `LQ` algorithm we replaced the function `sort` by a series of instructions, with the help of which the computing time of the secondary loop becomes $\Theta(n)$.

Having replaced the function `sort` the secondary loop of the algorithm (steps 5-13) is now executed in time $\Theta(n)$.

Conclusion.

In this paper we have presented an improvement of Lagrange's quadratic bound for computing upper bounds on the positive roots of polynomials.

The computing time of `LQ` is, of course, $O(n^2)$ but, as described in [4], the time expression has a second term; to wit, it is

$$\alpha n^2 + \Theta(n \cdot log(n)).$$

In this paper we reduce the computing time of `LQ` to

$$\alpha n^2 + \Theta(n).$$

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ACKNOWLEDGEMENTS: The work is partially supported by the Russian Fund for Basic Research (project № 16-07-00420).

Received 18 October 2016

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УДК 512.71
DOI: 10.20310/1810-0198-2016-21-6-1933-1937

О ВЫЧИСЛЕНИИ КВАДРАТИЧНОЙ ГРАНИЦЫ ЛАГРАНЖА ДЛЯ ПОЛОЖИТЕЛЬНЫХ КОРНЕЙ ПОЛИНОМА

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Лагранжева квадратичная оценка, для границы положительных корней многочленов состоит из двух частей. Сортировка одномерных массивов используется во второй части алгоритма Лагранжа во всех известных реализациях. Мы предлагаем изменить эту часть алгоритма так, чтобы исключить сортировку. В итоге сложность вычислений в этой части уменьшается с $O(n \cdot log(n))$ до $O(n)$.
Ключевые слова: квадратичная граница Лагранжа для корней; положительные корни полиномов

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БЛАГОДАРНОСТИ: Работа выполнена при финансовой поддержке Российского фонда фундаментальных исследований (проект № 16-07-00420).

Поступила в редакцию 18 октября 2016 г.

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Информация для цитирования:
Akritas A.G., Malaschonok G.I. On the computation of Lagrange's quadratic bound for positive roots of polynomials. *Vestnik Tambovskogo universiteta. Seriya Estestvennye i tekhnicheskie nauki – Tambov University Review. Series: Natural and Technical Sciences*, 2016, vol. 21, no. 6, pp. 1933-1937. DOI: 10.20310/1810-0198-2016-21-6-1933-1937