About Bruhat decomposition in domain

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Abstract. Present and discuss an algorithm for factoring polynomials of several variables. We discuss the implementation of algorithms in the system Mathpar.

Let R be a commutative domain, $A = (a_{i,j}) \in R^{n\times n}$ be a matrix of order n, $\alpha^k_{i,j}$ be $k \times k$ minor of matrix A which disposed in the rows $1, 2, .., k-1, i$ and columns $1, 2, .., k-1, j$ for all integers $i, j, k \in \{1, .., n\}$. We suppose that row i of matrix A is situated at the last row of the minor and column j of matrix A is situated at the last column of the minor. We denote $\alpha^0 = 1$ and $\alpha^k = \alpha^k_{k,k}$ for all diagonal minors $(1 \le k \le n)$. And we use notation δ_{ij} for Kronecker delta.

Theorem 1.
Let matrix $A = (a_{i,j})$ has rank $r \le n$ and $\alpha^s \ne 0$ for $s = 1, 2, .., r$ then matrix A is equals the following product of three matrices

$$A = (a^j_{i,j})(\delta_{ij}\alpha^{i-1}\alpha^i)^{-1}(a^i_{i,j}).$$

The matrix $L = (a^j_{i,j})$ is a low triangular matrix which has only r nonzero first columns, matrix $U = (a^i_{i,j})$ is an upper triangular matrix which has only r nonzero first rows, and $D = (\delta_{ij}\alpha^{i-1}\alpha^i)^{-1}$ is a diagonal matrix.

This theorem gives LDU decomposition of a matrix A in the domain. Let I be identity matrix and P be the matrix with second unit diagonal: $P^2 = I, P \ne I$.

Theorem 2.
Let $A = LDU$ be the LDU-decomposition of matrix A, and $A = PB$. Then

$$B = (PLP) \cdot (PD) \cdot U$$

is a Bruhat decomposition of matrix B in domain R.

If matrix $A = (a_{i,j})$ has rank $r \le n$ and $\alpha^s \ne 0$ for $s = 1, 2, .., r$ then matrix $V = (PLP$ is an upper triangular matrix which has only r nonzero last columns.

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We shell construct the LDU decomposition with the help of block recursive method.

Let $\mathcal{A}_n^k = (\alpha_{i,j}^{k+1})$ be the matrix of size $(n-k)\times(n-k)$ with elements $\alpha_{i,j}^{k+1}$, $i,j = k+1,..,n-1,n$, and $\mathcal{A}_n^0 = (\alpha_{i,j}^1) = A$.

We denote L_s^k, D_s^k and U_s^k the block of size $(s-k)\times(s-k)$, which stands at the intersection of rows $k+1,..,s$ and columns $k+1,..,s$ of matrices L, D and U, correspondingly.

We shell write LDU decomposition in block form

$$A = LDU = \begin{pmatrix} L_k^0 & 0 \\ L_{0,k}^{k,n} & L_n^k \end{pmatrix} \begin{pmatrix} D_k^0 & 0 \\ 0 & D_n^k \end{pmatrix} \begin{pmatrix} U_k^0 & U_{k,n}^{0,k} \\ 0 & U_n^k \end{pmatrix}$$

$$\mathcal{A}_n^k = L_n^k D_n^k U_n^k = \begin{pmatrix} L_s^k & 0 \\ L_{k,s}^{s,n} & L_n^s \end{pmatrix} \begin{pmatrix} D_s^k & 0 \\ 0 & D_n^s \end{pmatrix} \begin{pmatrix} U_s^k & U_{s,n}^{k,s} \\ 0 & U_n^s \end{pmatrix}$$

Algorithm.

Input data is a matrix $\mathcal{A}_n^k$ and determinant α^k ($\alpha^0 = 1, 0 \le k < n$). Output data

$L = (a_{i,j}^j)$, $D = (\delta_{ij}\alpha^i\alpha^{i-1})^{-1})$, $U = (a_{i,j}^i)$.

The integers k ans s may take arbitrary values that satisfy inequalities $0 \le k < s \le n$. Since $A = A_n^0$ and $\alpha^0 = 1$, this algorithm computes the decomposition of matrix A: $A = LDU$.

ALGORITHM LDU

Input: $(\mathcal{A}_n^k, \alpha^k)$, $0 \le k < n$.
Output: $\{\ L_n^k, \{\alpha^{k+1}, \alpha^{k+2}, .., \alpha^n\}, U_n^k, (L_n^k D_n^k)^{-1},\ (D_n^k U_n^k)^{-1}\ \}$

1. If $k = n-1$, $\mathcal{A}_n^{n-1} = (a^n)$ is a matrix of the first order, then we obtaint $\{a^n, \{a^n\}, a^n, 1, 1\}$.

2. If $k = n-2$, $\mathcal{A}_n^{n-2} = \begin{pmatrix} \alpha^{n-1} & \beta \\ \gamma & \delta \end{pmatrix}$ is a matrix of second order, then we obtain

$$\left\{ \begin{pmatrix} \alpha^{n-1} & 0 \\ \gamma & \alpha^n \end{pmatrix}, \{\alpha^{n-1}, \alpha^n\}, \begin{pmatrix} \alpha^{n-1} & \beta \\ 0 & \alpha^n \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -\gamma & \alpha^{n-1} \end{pmatrix}, \begin{pmatrix} 1 & -\beta \\ 0 & \alpha^{n-1} \end{pmatrix} \right\}$$

where $\alpha^n = (\alpha^k)^{-1} \begin{vmatrix} \alpha^{n-1} & \beta \\ \gamma & \delta \end{vmatrix}$

3. If the order of the matrix $\mathcal{A}_n^k$ more than two ($k < n-2$), then we choose an integer s in the interval ($k < s < n$) and divide the matrix into blocks

$$\mathcal{A}_n^k = \begin{pmatrix} \mathcal{A}_s^k & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}.$$

3.1. Recursive step

$\{L_s^k, \{\alpha^{k+1}, \alpha^{k+2}, .., \alpha^s\}, U_s^k, (L_s^k D_s^k)^{-1},\ (D_s^k U_s^k)^{-1}\} = \mathbf{LDU}(\mathcal{A}_s^k, \alpha^k)$

3.2. We comput

$$U_{s,n}^{k,s} = (D_s^k U_s^k)^{-1}\mathbf{B}, \quad L_{k,s}^{s,n} = \mathbf{C}(L_s^k D_s^k)^{-1},$$

$$\mathcal{A}_n^s = (\alpha^k)^{-1}\alpha^s(\mathbf{D} - \mathbf{C}(D_s^k U_s^k)^{-1} D_s^k U_{s,n}^{k,s}).$$

3.3. Recursive step

$$\{L_n^s, \{\alpha^{s+1}, \alpha^{s+2}, .., \alpha^n\}, U_n^s, (L_n^s D_n^s)^{-1},\ (D_n^s U_n^s)^{-1}\} = \mathbf{LDU}(\mathcal{A}_n^s, \alpha^s)$$

3.4 *Result:*

$$\{L_n^k, \{\alpha^{k+1}, \alpha^{k+2}, .., \alpha^n\}, U_n^k, (L_n^k D_n^k)^{-1},\ (D_n^k U_n^k)^{-1}\},$$

where

$$L_n^k = \begin{pmatrix} L_s^k & 0 \\ L_{k,s}^{s,n} & L_n^s \end{pmatrix}, D_n^k = \mathrm{diag}\{\alpha^k\alpha^{k-1}, .., \alpha^n\alpha^{n-1}\}, U_n^k = \begin{pmatrix} U_s^k & U_{s,n}^{k,s} \\ 0 & U_n^s \end{pmatrix}$$

$$(L_n^k D_n^k)^{-1} = \begin{pmatrix} (L_s^k D_s^k)^{-1} & 0 \\ Lj_{k,s}^{s,n} & (L_n^s D_n^s)^{-1} \end{pmatrix}, (D_n^k U_n^k)^{-1} = \begin{pmatrix} (D_s^k U_s^k)^{-1} & Uj_{s,n}^{k,s} \\ 0 & (D_n^s U_n^s)^{-1} \end{pmatrix}$$

$$Lj_{k,s}^{s,n} = -(L_n^s D_n^s)^{-1} L_{k,s}^{s,n} (L_s^k D_s^k)^{-1},\quad Uj_{s,n}^{k,s} = -(D_s^k U_s^k)^{-1} U_{s,n}^{k,s} (D_n^s U_n^s)^{-1}.$$

References

[1] *Grigoriev D.* Analogy of Bruhat decomposition for the closure of a cone of Chevalley group of a classical serie // Soviet Math. Dokl. Vol. 23. N. 2. 1981. P 393–397.

[2] *Malaschonok G.I.* Effective Matrix Methods in Commutative Domains // Formal Power Series and Algebraic Combinatorics. Springer, Berlin. 2000. P. 506-517.

[3] *Malaschonok G. I.* Fast Generalized Bruhat Decomposition // Computer Algebra in Scientific Computing, LNCS 6244, Springer, Berlin. 2010. P. 194-202.

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