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Loutra, Agia Paraskevi, Greece
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III. Mathematical Partnership for a subjects:

1. Physics,
2. Astronomy
3. Chemistry,
4. Biology,
5. Medicine,
6. Primary Mathematics,
7. Algebra, Geometry,
8. Mathematical Analysis.

Contents

New Functions of MathPartner in 2015

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Abstract

The report focuses on the changes that have occurred over the last year in the cloud mathematical project MathPartner. This includes Boolean functions, functions of set theory, additional special functions, new 3D graphics, new parallel interface, and significantly improved sidebar in the user interface.

One of the main fitures of Math Partner is the environment of working place. The definition of any mathematical object involves the definition of some environment, that is, the algebraic structure which contains this object. This structure is defined by numeric sets, algebraic operations and variable names. Examples: $R64[\,]$, $R[x]$, $Q[x,y]$ or $R_{max,+}[t]$.

However, Boolean algebra is available to the user in any environment. Operations of conjunction, disjunction, and negation are performed on the variables **true** and **false**. But there are no special characters for **true** and **false**. Zero is considered to be a **false** and any not zero element is the **true**. All operations of comparison of mathematical objects return values 0 and 1 with a numerical type $Z64$.

These values can be used in the Boolean operations or in the branching statements (*for, if-else, while-do*) of Matnpar language. Boolean operations may be applied to matrices. In this case conjunction and disjunction operations do similar to the matrix * and + operations. So it is very easy to calculate the composition of relations as a conjunction of the relevant matrices.

The user is now available algebra of subsets of real numbers with operations: $\cup$ (union of sets), $\cap$ (intersection of sets), $\setminus$ (set minus), $\triangle$ (set symmetric minus), s' (complement to the set s).

They can use notations for such subsets: $\backslash(a,b\backslash)$ â open interval, $\backslash[a,b\backslash]$ â closed interval, $\backslash(a,b\backslash]$ â half-open interval, $\backslash[a,b\backslash)$ â half-closed interval, $\emptyset$ â empty set.

The list special functions, with which you can work, both in symbolic and numeric form additionally includes several new functions.

Bessel functions:

$$BesselJ(n,x) \text{ and } BesselY(n,x).$$

Legendre polynomials:

$$LegendreP(n,x) = 1/(2n)\sum_{k=0}^{n}\binom{n}{k}^2(x-1)^{n-k}(x+1)^k.$$

Associated Legendre polynomials:

$$LegendreP(n,m,x) = P_n^m(x) = (-1)^m(1-x^2)^{m/2}(d/dx^m)P(n,x).$$

Spherical Harmonic:

$$\begin{gathered}sphericalHarmonic(n,m,\theta,\phi) = \\ [LegendreP(n,m,\cos(\theta))\cos(m\phi), \\ LegendreP(n,m,\cos(\theta))\sin(m\phi)].\end{gathered}$$

which is solution of Laplace's equation in spherical coordinates.

Spherical function:

$$sphericalHarmonicR(n,m,r,\theta,\phi) = r^nY(n,m,\theta,\phi)$$

Spherical Harmonic in Cartesian coordinates:

$$sphericalHarmonicCart(n,m,x,y,z) = Y(n,m,\theta,\phi)$$

for $r = \sqrt{(x^2+y^2+z^2)}$, $z = r\cos(\theta)$, $x = r\sin(\theta)\cos(\phi)$, $y = r\sin(\theta)\sin(\phi)$

Spherical function in Cartesian coordinates:

$$\begin{gathered}sphericalHarmonicRCart(n,m,x,y,z) \\ = r^n sphericalHarmonicCart(n,m,x,y,z)\end{gathered}$$

In the report, we discuss also the new opportunities of constructing images of surfaces, a new parallel interface, and significantly improved sidebar in user interface.

New address of the mirror with Math Partner see at [4]. Here we have today 16G memory and 16 kernels.

Acknowledgments

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An Improvement on Lagrange's Quadratic Bound on the Values of the Positive Roots of Polynomials

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Abstract

Lagrange's[1] quadratic bound, `LQ`, on the values of the positive roots of polynomials consists of two parts.

In the first part of `LQ` a list is constructed containing sub-lists, which correspond to the coefficients of the given polynomial. The sub-lists corresponding to positive coefficients remain empty throughout the execution of the algorithm. By contrast, the sub-lists corresponding to negative coefficients of the polynomial may end up containing 0, 1, 2 or more radicals of the preceding positive coefficients.

In the second part of `LQ`, the function `sort` is applied to all sublists containing more than 2 radicals and the sum of the largest two radicals in each sub-list is a possible candidate to be returned as the bound.

The computing time of the second part as is currently implemented is $O(n \cdot log(n))$. With our improvement we reduce the computing time of the second step of `LQ` to $O(n)$.

Introduction

On p. 553 of his original paper [3][2] — or on p. 32, of his famous book [4], which constitutes the 8-th volume of *Œuvres de Lagrange*, edited by Joseph Alfred Serret [5] — Lagrange only states that given the polynomial F, where

$$-\mu y^{r-m} - \nu y^{r-n} - \bar{\omega} y^{r-p} - \cdots$$

are its "negative terms", then an upper bound for the real roots of F is given by the sum of the first two largest of the quantities

$$\sqrt[m]{\mu},\ \sqrt[n]{\nu},\ \sqrt[p]{\bar{\omega}}, \ldots$$

[1] Italian mathematician Joseph-Louis Lagrange, born Giuseppe Lodovico Lagrangia, (25 January 1736 - 10 April 1813).

[2] Presented to the Berlin Academy on April 20, 1769.

or "a number larger than this sum."

The interesting history of this theorem can be found elsewhere [1]. Here we simply state the theorem without its proof.

Theorem 1 (Lagrange, 1767) *Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0,$, be a non constant monic polynomial of degree n over $\mathbb{R}$ and let $a_{n-j} : j \in J$ be the set of its negative coefficients. Then an upper bound for the positive real roots of f is given by the sum of the largest and the second largest members in the set $\left\{ \sqrt[j]{|a_{n-j}|} : j \in J \right\}$. That is,*

$$b = \max_{\{a_{n-i}, a_{n-k} \in J\}} (\sqrt[i]{|-a_{n-i}|} + \sqrt[k]{|-a_{n-k}|}). \tag{1}$$

Lagrange's Quadratic Algorithm `LQ`

In `LQ` we use the list (of lists) t of length $n+1$, in which initially each entry is [ ], the empty list. The list $t_{n-j} = t[n-j]$ corresponds to the coefficient a_{n-j} of the polynomial and, if $a_{n-j} > 0$, then the list $t[n-j]$ contains all the minimum values produced by $a_{n-j} > 0$ when "paired" with various negative coefficients a_{n-i}, with $i > j$.

The algorithm works as follows:

- *each* negative coefficient a_{n-i} of the polynomial is "paired" with *each one* the preceding positive coefficients a_{n-j}, $(i > j)$ and the minimum is taken of all the radicals of the form

$$\sqrt[i-j]{-a_{n-i}/a_{n-j}} \tag{2}$$

  as indicated in Lagrange's theorem (Theorem 1); each minimum is then appended to the corresponding list $t[n-j]$,

- we initialize a temporary bound to 0, and then for each non-empty list $t[n-j]$ we proceed as follows: (a) if the list $t[n-j]$ has a single element, and its value is greater than the temporary bound, then *it* (the single element) becomes the temporary bound and, (b) if the list $t[n-j]$ has more than one element, we sort them in increasing order and take the sum of the largest two; if the sum is greater than the temporary bound, then *it* (the sum) becomes the temporary bound; at the end the temporary bound is taken as the estimate of the bound.

An algorithmic description of Lagrange's quadratic method `LQ` is presented in Algorithm 1 below.

Algorithm 1: `LQ(f, x)`, Lagrange's Quadratic Algorithm.

Input: A univariate polynomial
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$, with $a_n > 0$.
Output: An upper bound on the values of the positive roots of $f(x)$.

```
// at least one sign variation (v ≥ 1)?
cl ← [a_0, a_1, a_2, ..., a_{n-1}, a_n];          /* list of length n+1 */
v ← number of sign variations in cl;
if v = 0 then return 0;
;
// initialize variables
m ← length(cl);
t ← [ [ ], [ ], [ ], ..., [ ], [ ] ];             /* list of length n+1 */
// (1) main loop, the same loop in the improved algorithm 2
1 for j = 0 to m − 1 step 1 do
2     if cl(j) < 0 then
          b ← +∞;
          index ← m;
          for k = j + 1 to m − 1 step 1 do
3             if cl(k) > 0 then
                  q ← (−cl[j]/cl[k])^{1/(k−j)};
4                 if q < b then
                      b ← q;
                      index ← k;
                  end
              end
          end
          t[index] ← append(t[index], b);
      end
  end
// (2) secondary loop to process the list of lists t
b ← 0;
5 for j = 0 to m − 1 step 1 do
      tp ← t[j];
6     if tp ≠ [] then
7         if length(tp) = 1 then
              tp ← tp[0];                  /* enumeration starts from 0 */
          else
              tp ← sort(tp);             /* sort tp in increasing order */
              tp ← sum(tp[−2 :]);   /* sum of the largest two values */
          end
8         if tp > b then
              b ← tp;
          end
      end
  end
  return b
```

Algorithm 2: `LQ(f, x)`, Lagrange's Quadratic Algorithm **Improved**.

Input: A univariate polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$, with $a_n > 0$.
Output: An upper bound on the values of the positive roots of $f(x)$.

```
   // at least one sign variation (v ≥ 1)?
   cl ← [a_0, a_1, a_2, ..., a_{n-1}, a_n];                    /* list of length n + 1 */
   v ← number of sign variations in cl;
   if v = 0 then return 0;
   ;
   // initialize variables
   m ← length(cl);
   t ← [ [ ], [ ], [ ], ..., [ ], [ ] ];                       /* list of length n + 1 */
   // (1) main loop, the same loop in the prevoiuse algorithm 1
   // (2) secondary loop to process the list of lists t without sort
   b ← 0;
1  for j = 0 to m − 1 step 1 do
2      if tp ≠ [ ] then
3          tp ← t[j];
           ltp ← length(tp);
           if ltp = 1 then
               sc ← tp[0] ;                                  /* enumeration starts from 0 */
           end
           c ← [ ];
4          if ltp > 2 then
5              if tp[0] < tp[1] then
                   c ← c[tp[0], tp[1]];
               else
                   c ← c[tp[1], tp[0]];
               end
           else
               c ← c[tp[0], tp[1]];
           end
6          for k = 2 to ltp − 1 step1 do
7              if tp[k] > c[0] then
8                  if tp[k] > c[1] then
                       c[0] ← c[1];
                       c[1] ← tp[k];
                   else
                       c[0] ← tp[k];
                   end
               end
               k ← k + 1;
           end
           sc ← sum(c[−2 :]);                                /* sum the two values of c */
9          if sc > b then
               b ← sc;
           end
       end
   end
   return b
```

Note the function `sort` in step 7 of the Algorithm 1 above. Obviously, the computing time of the secondary loop (steps 5-8) is $O(n \cdot log(n))$.

Improved Lagrange's Quadratic Algorithm `LQ`

In the improved `LQ` algorithm we replaced the function `sort` by a series of instructions, with the help of which the computing time of the secondary loop becomes $\Theta(n)$.

Having replaced the function `sort` the secondary loop of the algorithm (steps 5-13) is now executed in time $\Theta(n)$.

Conclusions

In this paper we have presented an improvement of Lagrange's quadratic bound for computing upper bounds on the positive roots of polynomials.

The computing time of `LQ` is, of course, $O(n^2)$ but, as described in [1], the time expression has a second term; to wit, it is

$$\alpha n^2 + \Theta(n \cdot log(n)).$$

In this paper we reduce the computing time of `LQ` to

$$\alpha n^2 + \Theta(n).$$

Acknowledgments

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[6] Stedall, J.: *From Cardano's great art to Lagrange's reflections: filling a gap in the history of algebra*; European Mathematical Society, (2011).

[3] The digitized form of *Œuvres de Lagrange* [5], Lagrange's collected work, can be found in the sites `http://sites.mathdoc.fr/OEUVRES/`, and `https://gdz.sub.uni-goettingen.de/`. Following Stedall's remark ([6], p. 209), Lagrange's article is listed here with two dates: the first one is the year when the paper is known to have been written — as recorded by Lagrange himself ([2], p. 384) — and the other one is the year in which the volume of papers was published.

Block-recursive algorithm LU-decomposition of matrices over idempotent semifields

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Abstract

This report is dedicated to block-recursive algorithm LU-decomposition of matrices over idempotent semifields. We formulate an algorithm and give an example of its use.

Let R – idempotent semifield, $A^{'} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $A^{'} \in R^{n \times n}$. We find the decomposition $A^{'} = LU$.

1. We expand the block $A = L_1 U_1$. If does not exist decomposition of A, then the LU-decomposition does not exist for the entire matrix.

2. We will solve the matrix equation $L_1 U_2 = B$ and $L_3 U_1 = C$. If the solution does not exist for at least one equation, the LU-decomposition does not exist for the matrix $A^{'}$.

The solution to these matrix equations can be the set of matrices. Let U_2^m, L_3^m – matrix with the highest values in the "floating" elements solutions. U_2^0, L_3^0 – matrix with the smallest possible values, that is with algebraic zeros on the "floating" elements solutions.

3. If there is a decomposition of block D: $D = L_4 U_4$.

a) if $L_3^m U_2^m \leq_T D$ (we must comply with all the relevant elements of inequality), then $L = \begin{pmatrix} L_1 & \mathbb{0} \\ L_3^m & L_4 \end{pmatrix}$, $U = \begin{pmatrix} U_1 & U_2^m \\ \mathbb{0} & U_4 \end{pmatrix}$;

else b) if $L_3^0 U_2^0 \leq_T D$, then $L = \begin{pmatrix} L_1 & \mathbb{0} \\ L_3^0 & L_4 \end{pmatrix}$, $U = \begin{pmatrix} U_1 & U_2^0 \\ \mathbb{0} & U_4 \end{pmatrix}$;

else c) decomposition does not exist.

4. If LU-decomposition does not exist for D, we must solve the following problem:

$$L_3 U_2 \oplus D^{'} = D.$$

Among the many solutions of this matrix equation, we must choose the decomposable matrix: $D^{'} = L_4^{'} U_4^{'}$.

Then the decomposition — $L = \begin{pmatrix} L_1 & \mathbb{0} \\ L_3 & L_4^{'} \end{pmatrix}$, $U = \begin{pmatrix} U_1 & U_2 \\ \mathbb{0} & U_4^{'} \end{pmatrix}$.
If the matrix equation has no solution, or among the solutions is not decomposable matrix, decomposition $A^{'} = LU$ does not exist.

Example. $max, +$

$A^{'} = \begin{pmatrix} 2 & 4 & 3 & 7 \\ 3 & 5 & 6 & 8 \\ 2 & 4 & 5 & 7 \\ 1 & 3 & 3 & 9 \end{pmatrix}$. $L_1 = \begin{pmatrix} 0 & \mathbb{0} \\ 1 & 0 \end{pmatrix}$, $U_1 = \begin{pmatrix} 2 & 4 \\ \mathbb{0} & 5 \end{pmatrix}$.

$L_1U_2 = B$; $\begin{pmatrix} 0 & \mathbb{0} \\ 1 & 0 \end{pmatrix} U_2 = \begin{pmatrix} 3 & 7 \\ 6 & 8 \end{pmatrix}$; $U_2 = \begin{pmatrix} 3 & 7 \\ 6 & [\mathbb{0}, 8] \end{pmatrix}$;

$U_2^m = \begin{pmatrix} 3 & 7 \\ 6 & 8 \end{pmatrix}$; $U_2^0 = \begin{pmatrix} 3 & 7 \\ 6 & \mathbb{0} \end{pmatrix}$;

$L_3U_1 = C$; $L_3 = \begin{pmatrix} 0 & [\mathbb{0}, -1] \\ -1 & [\mathbb{0}, -2] \end{pmatrix}$; $L_3^m = \begin{pmatrix} 0 & -1 \\ -1 & -2 \end{pmatrix}$; $L_3^0 = \begin{pmatrix} 0 & \mathbb{0} \\ -1 & \mathbb{0} \end{pmatrix}$;

D – decomposable. $L_4 = \begin{pmatrix} 0 & \mathbb{0} \\ -2 & 0 \end{pmatrix}$, $U_4 = \begin{pmatrix} 5 & 7 \\ \mathbb{0} & 9 \end{pmatrix}$.

$L_3^m U_2^m \not\leq_T D$; $\begin{pmatrix} 5 & 7 \\ 4 & 6 \end{pmatrix} \not\leq_T \begin{pmatrix} 5 & 7 \\ 3 & 9 \end{pmatrix}$;

$L_3^0 U_2^0 \leq_T D$; $\begin{pmatrix} 3 & 7 \\ 2 & 6 \end{pmatrix} \leq_T \begin{pmatrix} 5 & 7 \\ 3 & 9 \end{pmatrix}$;

Decomposition $L = \begin{pmatrix} L_1 & \mathbb{0} \\ L_3^0 & L_4 \end{pmatrix}$, $U = \begin{pmatrix} U_1 & U_2^0 \\ \mathbb{0} & U_4 \end{pmatrix}$.

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A differential equation with delayed argument and application of Laplace transform

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Abstract

Applications of the Laplace transform for solving differential equations method are well known. It permits to reduce an infinitesimal problem to an algebraic one that may be solved symbolically or symbolic-numerically. We suggest the usage of series expansion of some kind for symbolic-numerical solution with a necessary accuracy. It extends the class of equations to be solved by this method.

Consider an equation

$$x^{(n)}(t) + \sum_{j=1}^{n} \sum_{k=0}^{N} a_{jk} x^{(n-j)}(t - t_k) = f(t), \tag{1}$$

with initial conditions $x^{(n-j)}(0) = x_0^{(n-j)}, j = 1, \ldots, n$, where $0 < t_k < t_{k+1} < T, k = 0, \ldots, N$. All functions are composite on the segment $\mathbf{T}$: $0 \le t \le T$, their components are represented as finite sums of exponents with polynomial coefficients. It permits to write symbolically the Laplace image of the equation (1):

$$\left(p^n + \sum_{j=1}^{n} \sum_{k=0}^{N} a_{jk} e^{-pt_k} p^{n-j} \right) X(p) =$$

$$\sum_{j=1}^{n} p^{j-1} x_0^{(n-j)} + \sum_{j=1}^{n-1} \sum_{k=0}^{N} a_{jk} p^{j-1} x_0^{(n-j)} e^{-pt_k} + F(p),$$

where $X(p)$ and $F(p)$ are the Laplace images of $x(t)$ and $f(t)$, correspondingly, and $F(p)$ is a sum of exponents with polynomial coefficients.

Denote

$$Q(p) = \sum_{j=1}^{n} p^{j-1} x_0^{(n-j)} + \sum_{j=1}^{n-1} \sum_{k=0}^{N} a_{jk} p^{j-1} x_0^{(n-j)} e^{-pt_k} + F(p),$$

$$D(p) = p^n + \sum_{j=1}^{n}\sum_{k=0}^{N} a_{jk} e^{-pt_k} p^{n-j}, \quad \text{and} \quad X(p) = \frac{Q(p)}{D(p)}.$$

The last step of the algorithm is the Inverse Laplace transform. We must find a half-plane to chose a vertical line for inverse transform and for a series expansion.

Writing t_k as $t_k = \frac{\tau_k}{\sigma_k}$, denote $\sigma = LCM_k(\sigma_k)$, and $t_k = \frac{\tilde{\tau}_k}{\sigma}$.

Denote $e^{-\frac{p}{\sigma}} = z$. Then

$$X(p) = \frac{\sum_{j=1}^{n} p^{j-1} x_0^{(n-j)} + \sum_{j=1}^{n-1}\sum_{k=0}^{N} a_{jk} p^{j-1} x_0^{(n-j)} z^{\tilde{\tau}_k} + F(p)}{p^n + \sum_{j=1}^{n}\sum_{k=0}^{N} a_{jk} z^{\tilde{\tau}_k} p^{n-j}}. \quad (2)$$

Substituting $e^{-\frac{p}{\sigma}}$ instead of z, we obtain the series for $X(p)$ by $e^{-\frac{np}{\sigma}}$, which converges in some neighbourhood of ∞:

$$\sum_n A_n e^{-\frac{np}{\sigma}}, \quad (3)$$

where A_n are proper fractions, and can be represented as sums of partial fractions.

For the series (3) the Inverse Laplace transform may be written symbolically.

A problem is to define n and Re p sufficient for designed accuracy of the differential equation.

Let us take the $n-th$ Taylor approximation of $X(p)$ and find its inverse Laplace image. Denote by $\tilde{x}(t)$ an approximate solution of (1), which is equal to this image. The accuracy of such solution we denote by ϵ, i.e.

$$\max_{\mathbf{T}} |x(t) - \tilde{x}(t)| < \epsilon. \quad (4)$$

The remainder term of (3) may be written in the form $\sum_{k=n} \frac{\alpha_k}{p^k} e^{-\frac{kp}{\sigma}}$. Demand

$$|p| \frac{|\alpha_n|}{(\text{Re } p)^n} e^{-\frac{n\text{Re } p}{\sigma}} < \epsilon.$$

Then we obtain (4) for each $t \in \mathbf{T}$.

The method is suitable for systems of differential equations of the kind described above.

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Description of two qubit entanglement space in terms of polynomial invariants: a challenge for computer algebra

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Abstract

We consider computational aspects of characterising the entanglement space of two qubit mixed states via the polynomial invariants of local unitary group $SU(2) \times SU(2)$. Although in the literature a number of computer algebra based algorithms has been designed for construction of the ring of invariant polynomials, the underlying computations related to two qubits are too hard for those algorithms. In the talk we restrict ourselves by a subset of the two qubit states containing so-called $X-$states and investigate its invariant ring.

Introduction

The entanglement of qubits (quantum bits) provided by their quantum correlations is the main resource [1] of quantum computing and quantum information processes, e.g., superdense coding, teleportation and cryptography. By this reason a qualitative and quantative characterization of entanglement is a topical research problem.

Problem statement

A two qubit state is described by a 4×4 Hermitian, semi-positive and unit-trace matrix of the (so-called Fano) form

$$\varrho = \frac{1}{4}\left[I_2 \otimes I_2 + \sum_{i=1}^{3} a_i \sigma_i \otimes I_2 + \sum_{i=1}^{3} b_i I_2 \otimes \sigma_i + \sum_{i,j=1}^{3} c_{ij} \sigma_i \otimes \sigma_j\right]$$

where σ_i are the Pauli matrices, I_2 is the identity 2×2 matrix and a_i, b_i, c_{ij} are real numbers. The entanglement of a quantum state is invariant under

the adjoint action of the local unitary group $(g, \varrho) \to g\varrho g^\dagger$, $g \in G := SU(2) \times SU(2)$. These transformations of ϱ induce transformations on the 15-dimensional real space

$$W := \{ (a_i, b_j, c_{kl}) \in \mathbb{R}^{15} \mid i, j, k, l = 1, 2, 3 \} .$$

It follows that the corresponding $G-$invariant polynomials accumulate all relevant information on the two qubit entanglement.

The ring $\mathbb{R}[W]^G$ of $G-$invariant polynomials is Cohen-Macaulay [2] and the *research problem of computer algebra* is to construct the Hironaka decomposition

$$\mathbb{R}[W]^G = \bigoplus_{f_k \in F_s} f_k \, \mathbb{R}[F_p] .$$

Here F_p is a set of homogeneous and algebraically independent *primary invariants* and F_s is a set of homogeneous *secondary invariants*.

Computational issues

Based on the Gröbner bases techniques, several algorithms has been suggested and implemented in computer algebra software (cf. book [2] and its bibliography) to construct invariant rings for linear reductive groups. According to the literature, among those algorithms the Derksen's one, implemented in MAGMA and SINGULAR, is the most efficient. However, the invariant ring $\mathbb{R}[W]^G$ is computationally intractable by the last algorithm as well as by the other known algorithms.

Nevertheless, both sets F_p of primary and F_s of secondary invariants for $\mathbb{R}[W]^G$ were constructed in [3]. The authors of this paper exploited the following facts: (i) the knowledge on cardinalities of F_p (10) and F_s (15) and the degrees of their elements provided by the generating Molien function of G (Hilbert series in [2]); (ii) the results of paper [4] where a set F of 20 fundamental invariants in $\mathbb{R}[W]^G$ was presented. An irredundant set of invariants is called *fundamental* if it generates $\mathbb{R}[W]^G$. In so doing, the special features of $SU(2) \times SU(2)$ were used in construction of F_p and F_s whereas the computer algebra software (MAPLE) was applied to compute the ideal I of *syzygies* (*nontrivial algebraic relations*) generated by the fundamental invariants. The knowledge of syzygy ideal I is necessary because of the ring isomorphism

$$\mathbb{R}[W]^G \cong \mathbb{R}[y_1, \ldots, y_{20}]/I_F .$$

It should be emphasized that at the present time even verification of the algebraic independence for the 10 primary invariants of paper [3] cannot

be performed by means of computer algebra since the underlying Gröbner basis computation is too cumbersome.

$X-$states

Recently, we studied [5] the subspace of two qubit mixed states such that the density matrix reads

$$\varrho_X := \begin{pmatrix} \varrho_{11} & 0 & 0 & \varrho_{14} \\ 0 & \varrho_{22} & \varrho_{23} & 0 \\ 0 & \varrho_{32} & \varrho_{33} & 0 \\ \varrho_{41} & 0 & 0 & \varrho_{44} \end{pmatrix} .$$

These states got name $X-$states due to the visual similarity of the density matrix, whose non-zero entries lie only on the main and minor (secondary) diagonals, with the Latin letter "X":

Comparison with the above given Fano decomposition shows that the $X-$states belong to the 7-dimensional subspace W_X of the vector space W defined as:

$$W_X := \{\, w \in W \mid c_{13} = c_{23} = c_{31} = c_{32} = 0\,, a_i = b_i = 0\,,\ i = 1, 2\,\}\,.$$

Our interest to this subspace of a generic two qubit space is due to fact that many well-known states, e.g. the Bell states, Werner states, isotropic states and maximally entangled mixed states are particular subsets of the $X-$states. Since their introduction in 2007 many interesting properties of $X-$states have been established. Particularly, it was shown that for a fixed set of eigenvalues the states of maximal concurrence, negativity or relative entropy of entanglement are the $X-$states.

The restriction of the invariant polynomial ring $\mathbb{R}[W]^G$ to the invariant polynomial ring $\mathbb{R}[W_X]^{G_X}$ where G_X is a group of local unitary transformations that preserve the shape of $X-$states shows [5] that

$$G_X := SO(2) \times SO(2) \cong U(1) \times U(1) \subset SU(2) \times SU(2)\,,$$

and there is an injective homomorphism of the ring $\mathbb{R}[W]^G$ to the ring $\mathbb{R}[W_X]^{SO(2)\times SO(2)}$ which is freely generated.

Conclusions

A two qubit system is the simplest nontrivial quantum system which admits entangled states. However, even in this simplest case the rigorous and

comprehensive description of the entanglement in terms of local unitary polynomial invariants is an arduous computational challenge for modern computer algebra.

Acknowledgments

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Difference algebra aided discretization of quasilinear evolution equations

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Abstract

An algorithmic approach is considered to construct, on the Cartesian grids, finite difference approximations to quasilinear evolution equations $u_t+u_n+F(u,u_1,\ldots,u_{n-1})=0$ in dimension (1+1), where $u_i=\partial_x^i u$ $(i=1,\ldots,n)$ and F is a differential polynomial over the field $\mathbb{Q}(a_1,\ldots,a_m)$ of constants (parameters). The approach is based on the finite volume method in which the control volumes are rectangular boxes whose vertices are situated at grid points and the sides are parallel to the coordinate axes. In doing so, a finite difference approximation is obtained by combining of the numerical integration methods and the difference elimination of the spacial derivatives by construction of a Gröbner basis for the elimination ranking. The Gröbner basis is constructed by means of the Maple package *LDA* implementing the involutive algorithm. This approach is illustrated by the example of classical equation of the third order, the Korteveg-de Vries equation. The finite difference approximation generated for this example is applied to numerical solving of the initial value problem for solution of two-soliton type.

Research object

Let ∂_x be the derivation operator w.r.t. x and

$$\mathbb{R} := \mathbb{Q}(a_1,\ldots,a_i)\{u\}$$

be the ordinary differential polynomial ring over the (parametric) field of constants, $\mathbb{Q}(a_1, \ldots, a_i)$. Here we consider quasilinear evolution equations of the form

$$u_t = au_m + F(u_{m-1}, \ldots, u_1, u)\,, \quad 0 \neq a \in \mathbb{Q}\,, \quad m \in \mathbb{N}_{>0}\,, \tag{3}$$

where

$$u_k := \partial_x^k u\,, \quad (0 \leq k \leq m)\,, \quad u_0 := u\,,$$

and $F \in R$ is a differential polynomial of the order $m-1$ in ∂_x (denotation: $\mathrm{ord}(F) = m-1$) such that there is a differential polynomial $P \in \mathbb{R}$ satisfying

$$F = \partial_x P = \sum_{k=0}^{m-2} u_{k+1} \frac{\partial P}{\partial u_k}\,. \tag{4}$$

Given F, one can algorithmically verify whether or not such P exists and construct it in the case of existence. The equality (4) means that (3) admits the conservation law form

$$u_t = \partial_x \left(au_{m-1} + P\right)\,, \quad P \in \mathbb{R}\,, \quad \mathrm{ord}(P) = m-2\,. \tag{5}$$

The set of evolution equations admitting the polynomial conservation law (5) contains most of classical evolution equations, e.g., the Korteveg-de Vries (KdV) equation and KdV hierarchy, the Burgers equation and Burgers hierarchy, the Kuramoto-Sivashinsky equation, the Burgers-Huxley equation, etc., and their various generalizations (cf. [1]). All these equations have exact solutions that are useful in analysis of numerical methods for their solving.

Discretization method

To discretize equation (5), we follow the approach of paper [2] and convert the equation into the equivalent integral form

$$\oint_\Gamma \left(P + au_{m-1}\right) dt + u\, dx = 0\,, \tag{6}$$

where Γ is an arbitrary singly connected integration contour. Using the standard notation $u_j^n = u(t_n, x_j)$ for the grid function and the Cartesian grid with $t_{n+1} - t_n = \tau$, $x_{j+1} - x_j = h$ we choose the rectangular integration contour as a "control volume" (cf. [2]) and add $m-2$ integral relations

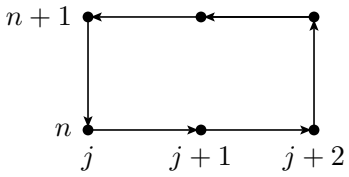

Figure 1: Basic integration contour

$$\int_{x_j}^{x_{j+1}} u_{k+1}\,dx = u_k(t,x_{j+1}) - u_k(t,x_j)\,, \quad k = 1,\ldots,m-2\,. \tag{7}$$

Now, to discretize (6) we apply a numerical evaluation method to the contour integral (6) in order to express it in terms of the grid functions and also (possibly different) numerical evaluation methods to the integrals in the left-hand sides of (7). Thereby, we obtain a system of difference equations containing $u_j^n, u_{1\,j}^{\ n}, \ldots, u_{m-1\,j}^{\ n}$. The last step in generation of a finite difference approximation (FDA) to (3) is algebraic elimination of the grid functions $u_{1\,j}^{\ n}, \ldots, u_{m-1\,j}^{\ n}$, which correspond to the proper partial derivatives of u, from the discrete system obtained. Such elimination can be done by means of the MAPLE package *LDA* [3, 4] which is freely available (`http://wwwb.math.rwth-aachen.de/Janet/`).

Example: FDA to the KdV equation

We illustrate the above described approach by example of the KdV equation

$$u_t + u_{xxx} + 6uu_x = 0\,. \tag{8}$$

Its integral conservation law form for the contour $\mathcal{C}$ of Figure 1 reads

$$\oint_{\mathcal{C}} (u_{xx} + 3u^2)\,dt + u\,dx = 0\,. \tag{9}$$

To approximate numerically the contour integral, we apply the trapezoidal rule to the integration over t as well as to the integration over x. For numerical approximations of the integral relations

$$\int_{x_j}^{x_{j+1}} u_{xx}\,dx = u_x(t,x_{j+1}) - u_x(t,x_j)\,, \quad \int_{x_j}^{x_{j+1}} u_x\,dx = u(t,x_{j+1}) - u(t,x_j)$$

we apply the trapezoidal rule for the integration of u_x and the midpoint rule for the integration of u_{xx}. This leads to the difference approximation

```
> with(LDA):
> L:=[(-(F(n,j)+F(n+1,j)-F(n,j+2)-F(n+1,j+2)) -
      (uxx(n,j)+uxx(n+1,j)-uxx(n,j+2)-uxx(n+1,j+2))-
      s2*(uxx(n,j)+uxx(n+1,j)-uxx(n,j+2)-uxx(n+1,j+2)))*tau/2+
      (u(n+1,j+1)-u(n,j+1))*2*h + s*(u(n+1,j+1)+u(n,j+1))*tau*h,
     (ux(n,j+1)+ux(n,j))*h/2-(u(n,j+1)-u(n,j)),
      2*uxx(n,j+1)*h-(ux(n,j+2)-ux(n,j))]:
> JanetBasis(L, [n,j], [uxx,ux,u,F],2);
```

$$\Big[\Big[(s2\,\tau+\tau)\,u(n+1,j+4)+(-2\,s2\,\tau-2\,\tau)\,u(n+1,j+3)+\left(2\,h^3\,s\,\tau+4\,h^3\right)u(n+1,j+2)+(2\,s2\,\tau+2\,\tau)\,u(n+1,j+1)+(-s2\,\tau-\tau)\,u(n+1,j)+(s2\,\tau+\tau)\,u(n,j+4)+(-2\,s2\,\tau-2\,\tau)\,u(n,j+3)+\left(2\,h^3\,s\,\tau-4\,h^3\right)u(n,j+2)+(2\,s2\,\tau+2\,\tau)\,u(n,j+1)-\tau\,s2\,u(n,j)-\tau\,u(n,j)+h^2\,\tau\,F(n+1,j+3)+h^2\,\tau\,F(n,j+3)-h^2\,\tau\,F(n+1,j+1)-h^2\,\tau\,F(n,j+1),\ h\,ux(n,j+1)+h\,ux(n,j)-2\,u(n,j+1)+2\,u(n,j),\ h^2\,uxx(n,j+1)-u(n,j+2)+2\,u(n,j+1)-u(n,j),\ \left(-h^2\,s2\,\tau-h^2\,\tau\right)uxx(n+1,j)-h^2\,\tau\,s2\,uxx(n,j)-h^2\,\tau\,uxx(n,j)+(s2\,\tau+\tau)\,u(n+1,j+3)+(-2\,s2\,\tau-2\,\tau)\,u(n+1,j+2)+\left(2\,h^3\,s\,\tau+4\,h^3+s2\,\tau+\tau\right)u(n+1,j+1)+(s2\,\tau+\tau)\,u(n,j+3)+(-2\,s2\,\tau-2\,\tau)\,u(n,j+2)+\left(2\,h^3\,s\,\tau-4\,h^3+s2\,\tau+\tau\right)u(n,j+1)+h^2\,\tau\,F(n+1,j+2)+h^2\,\tau\,F(n,j+2)-h^2\,\tau\,F(n+1,j)-h^2\,\tau\,F(n,j)\Big],\ [n,j],\ [uxx,\ ux,\ u,\ F]\Big]$$

```
> collect(%[1,1]/(4*tau*h**3),[s,s2,tau,h]);
```

$$\left(\frac{1}{2}\,u(n+1,j+2)+\frac{1}{2}\,u(n,j+2)\right)s+\frac{1}{h^3}\left(\left(\frac{1}{4}\,u(n+1,j+4)-\frac{1}{2}\,u(n+1,j+3)+\frac{1}{2}\,u(n+1,j+1)-\frac{1}{4}\,u(n+1,j)+\frac{1}{4}\,u(n,j+4)-\frac{1}{2}\,u(n,j+3)+\frac{1}{2}\,u(n,j+1)-\frac{1}{4}\,u(n,j)\right)s2\right)+\frac{\frac{1}{4}\,F(n+1,j+3)+\frac{1}{4}\,F(n,j+3)-\frac{1}{4}\,F(n+1,j+1)-\frac{1}{4}\,F(n,j+1)}{h}+\frac{1}{h^3}\left(\frac{1}{4}\,u(n+1,j+4)-\frac{1}{2}\,u(n+1,j+3)+\frac{1}{2}\,u(n+1,j+1)-\frac{1}{4}\,u(n+1,j)+\frac{1}{4}\,u(n,j+4)-\frac{1}{2}\,u(n,j+3)+\frac{1}{2}\,u(n,j+1)-\frac{1}{4}\,u(n,j)\right)+\frac{u(n+1,j+2)-u(n,j+2)}{\tau}$$

Figure 2: Construction of FDA to KdV with MAPLE

to (8) which is outputted by the following MAPLE code shown in Figure 2 with $P := 3u^2$.

The output is the left-hand side of the FDA to (8) written in the conventional form as

$$\frac{u_j^{n+1}-u_j^n}{\tau}+\frac{(P_{j+1}^{n+1}-P_{j-1}^{n+1})+(P_{j+1}^n-P_{j-1}^n)}{4h}+$$
$$+\frac{(u_{j+2}^{n+1}-2u_{j+1}^{n+1}+2u_{j-1}^{n+1}-u_{j-2}^{n+1})+(u_{j+2}^n-2u_{j+1}^n+2u_{j-1}^n-u_{j-2}^n)}{4h^3}=0\,.$$

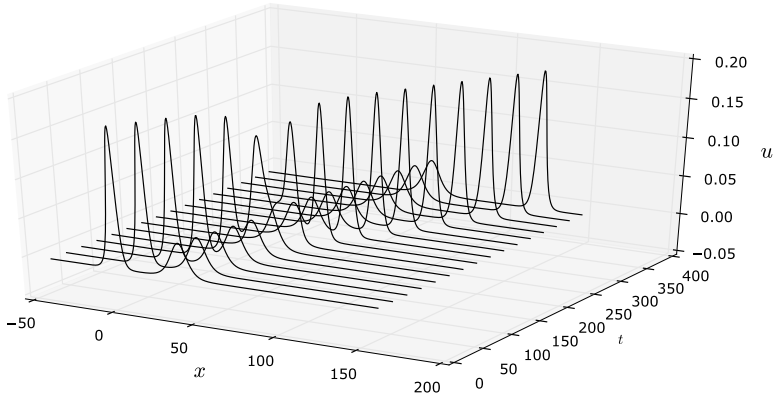

Figure 3: Dynamics of solution to KdV

Numerical solution

Since the obtained FDA to (8) has quadratic nonlinearity (due to $P = 3u^2$) in the grid function on the next time layer, in order to construct a numerical solution we use the following linearization

$$v_{k+1}^2 = v_{k+1}^2 - v_k^2 + v_k^2 = (v_{k+1} - v_k)(v_{k+1} + v_k) + v_k^2 \approx v_{k+1} \cdot 2v_k - v_k^2 \,.$$

By taking this linearization into account, we implemented a numerical procedure for construction of a solution to KdV in Python 2.7 freely downloadable from the Web page

`https://www.python.org/download/releases/2.7/`.

Figure 3 demonstrates the time evolution of numerical solution in the domain $x \in [0, 200]$ with $h = 0.4$ and $\tau = 0.2$ and for the initial value (Cauchy) problem with the initial data

$$u(t = 0) := f(x, 0, 0.4) + f(x - 20, 0, 0.2)$$

where

$$f(x, t, \kappa) := \frac{2\kappa^2}{\cosh^2\left[\kappa(x - 4\kappa^2 t)\right]}$$

is the exact one-soliton solution to (8).

Conclusions

The nonlinear evolution equations of the form we consider in this note are of a large interest in modern mathematical physics [1]. These equations belong the the class of so-called integrable equations which admit certain exact solutions. This fact allows to study modern numerical methods by their verification by the exact solutions. We use here our computer algebra aided approach to generation of finite difference approximations on Cartesian grids [2] and based on the difference Gröbner bases techniques implemented in MAPLE [3, 4]. As our numerical analysis of the KdV equations, the most famous and studied among the integrable evolution equations, shows, the discretization obtained by our methods has a good qualitative and quantative behavior.

As FIGURE 3 shows, the constructed numerical solution reveals a behavior inherent to localized solutions of KdV.

Acknowledgments

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Statistical analysis of awareness and knowledge of Alzheimers disease in Grenada

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Abstract

We use Wolfram Mathematica to analyze awareness and knowledge of Alzheimers disease (AD) in the Caribbean island of Grenada. AD is a devastating disease of aging for which there is currently no cure. Knowledge of AD is important as it helps families and healthcare workers seek and provide proper care. It also empowers people to address preventable risk factors. The basic goal of this study is to answer the question: Is Alzheimers disease knowledge the same between the SGU students, faculty, healthcare professionals and the non-SOM public?

Introduction

Alzheimer's disease (AD) is a devastating disease of aging for which there is currently no cure. There are clear racial differences in AD. For example, Blacks are 2-3X more likely than Whites to have AD, earlier onset, and are diagnosed in later stages of the disease [1]. These differences hold true for Caribbean Blacks whom also have increased rate of dementia diagnoses compared to the White population [2]. These disparities are not due to genetic factors but due to increased risk factors of cardiovascular disease, diabetes, missed diagnoses [3], and poorer AD knowledge in Blacks [4]. There is no data on AD incidence in Grenada. We presume that AD prevalence is high in Grenada due to AD risk factors of hypertension and diabetes being common diseases treated at the Grenada General Hospital.

The lack of AD data may be due to a lack of AD knowledge in Grenada. Knowledge of AD is important as it helps families seek proper care and healthcare workersmake accurate diagnoses. We endeavored to explore the

following aims: 1. Are Grenadians knowledgeable about AD? This addresses whether families recognize AD and seek help. 2. Are healthcare workers knowledgeable about AD? This addresses whether healthcare workers can recognize AD and make a diagnoses, an issue even in the USA. We used Wolfram Mathematica for basic statistical analysis of the survey on awareness and knowledge of AD in Grenada. Mathematica provides good facilities for statistics as well as good visualization tools.

Methods

IRB approval was via SGU and the Grenada Medical Association. Knowledge levels were investigated via a 30 item true/false Alzheimer's Disease Knowledge Scale (ADKS) survey administered to a total of 183 respondents The survey evaluates several domains of AD knowledge and calculates an ADKS score by summing correct responses. This survey includes the Shipley Institute of Living Scale which uses word matching of incresing difficulty as a measure of general intellectual functioning ([5]). This ADKS has been rigorously validated in a separate publication ([5]).

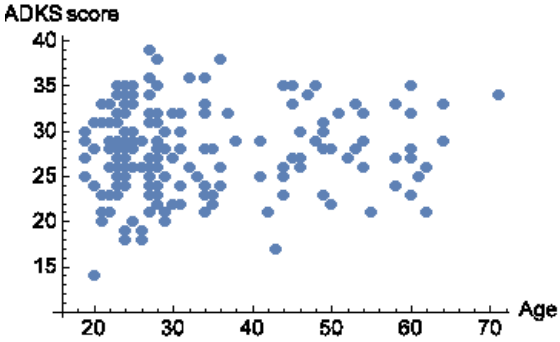

Figure 1: AD knowledge score is not affected by age, in contrast to studies in the US and Australia where ADKS increased with age [5,6].

Results

A large proportion of the respondents were black (41.5%), as the Grenadian population is primarily 82% Black, and 64.5% were born in Grenada. The

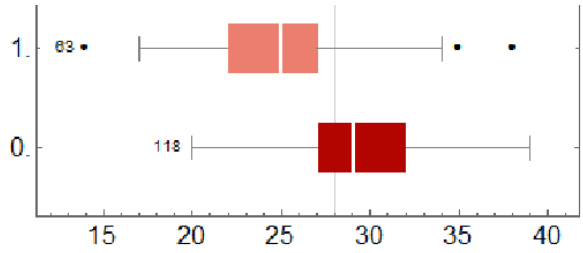

Figure 2: ADKS score was lower in Grenadians. 64.5% respondents were Grenadian born. (On the Figure, "1" corresponds to Grenadian born, "0" to non-Grenadian). The null hypothesis that the datasets have the same distribution is rejected at the 5 percent level based on the Pearson χ^2 test ($p = 6.3 \times 10^{-8}$).

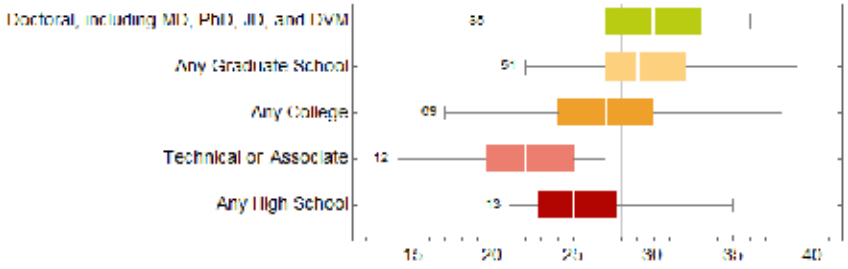

Figure 3: ADKS score increased with higher education. Percentages: High school (7.1%), Technical/Associate (6.6%), College (37.7%), Graduate School (27.9%), & Doctoral (19.1%).

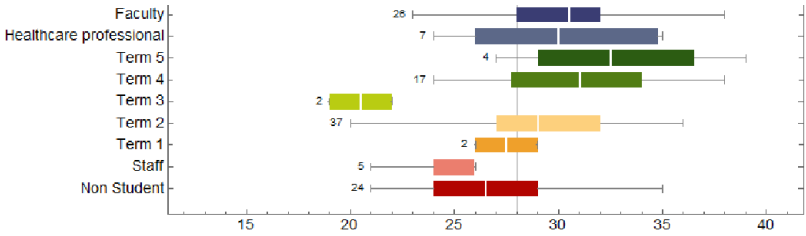

Figure 4: SOM affiliation affected ADKS score. Percentages: Faculty/Healthcare professional (17.8%), non-SOM (13.1%), Staff (2.7%), Terms 1-3 (22.4%), Terms 4/5 (11.5%).

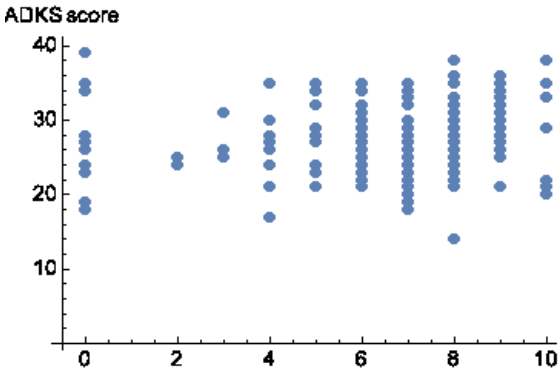

Figure 5: ADKS score tended to slightly increase with general intelligence. (Shipley Scale)

majority were young, and female, a similar observation as in a study from Australia [6]. The difference in ADKS between non-Grenadians and Grenadians (Fig 2) could be due to race, gender, education level or general intelligence. There was no significant difference between gender (32.8 ± 12.27; the null hypothesis that the datasets have the same distribution is not rejected at the 5 percent level based on the Pearson χ^2 test, $p = 0.33$), similar to findings in US and Australia [6,7]. We also did not find any trends for race, similar to Australia [6]. Higher levels of education led to increased ADKS score (Fig 3), which was similar to studies in the USA and Australia [6, 7]. Increased education in those affililiated with School of Medicine resulted in higher ADKS scores. Impressively, ADKS increased from students in lower to higher terms (Fig 4). Increased ADKS scores weakly correlated with performance in the general intelligence test (Shipley intelligence scale, Fig 5) . Knowledge in specific domains showed similar trends to that of US, and Australia [6,7]. However, Grenadians (GND) tended to score lower in Risk Factors, Caregiving, Symptoms and Course of disease (Table 1).

Conclusions

This study suggests that AD is likely to be diagnosed properly by health care providers. However, AD risk factors and symptoms may not be recognized and understood by the general Grenadian population. Outreach will raise awareness of AD, reduce risk factors, and will likely reduce the AD incidence in Grenada.

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Table 1: Knowledge in specific domains.

Domain	GND	Non GND	USA	Aust
Life Impact	0.80 (0.39)	0.80 (0.38)	0.70	0.87
Risk Factors	**0.55 (0.45)**	0.66 (0.43)	0.60	0.65
Symptoms	**0.66 (0.46)**	0.75 (0.43)	0.71	0.80
Treatment/ Management	0.82 (0.37)	0.87 (0.31)	0.80	0.86
Assessment	0.76 (0.42)	0.83 (0.35)	0.80	0.86
Caregiving	**0.64 (0.44)**	0.73 (0.42)	0.71	0.81
Course	**0.65 (0.44)**	0.77 (0.40)	0.71	0.75

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3D Printing and its Application in the Health Care Industry in Grenada, a Small Island Developing State

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Abstract

The emergence of 3D printing in the last decade has the potential to revolutionize both the cost and delivery of health care throughout the world, and in the context of this research Grenada, a small island developing state. In several areas of health care delivery and cost, it has been found that the use of 3D printing will not only lead to cheaper access but also to wider availability.

In this research we examined the areas in health care that will be affected by this new technology and how it will affect health care delivery in Grenada. This research states the challenges confronting the health care industry in Grenada and how this may be improved by the introduction of 3D printing. This was a partnership research involving professionals from both IT and medicine.

Introduction

3D printing is one way to meet the need for a faster and more cost effective way of creating products such as body parts, machinery, and other utility items. The authors show how Grenada can benefit from this technology particularly with respect to the healthcare industry. As a result of the global health crisis, medical institutions and experts have developed extensive research and investments to help curb some of the life threatening and fatal health issues. The use of 3D printing throughout the world has helped improve the ways in which medical functions are carried out.

Grenada, a small island developing state, has had its share of medical crisis. The authors have therefore decided to undertake a review of some of the ways that this Caribbean island can benefit from the technological development of 3D printing. We provide an insight on the impact of 3D

printing and the ability of experts in the medical field to carry out their traditional functions, and whether patients are better off as a result of this technology. Of particular interest to us are the amputees and how the technology can be used to better their lives after amputation. Although the first choices of assistance would be prevention, however for now we present a way for making the lives of those who have undergone limb amputation more comforting. 3D printing can be used for production of prosthesis legs and hands which will be more affordable for amputees. Prosthesis objects made by some other method are very expensive and are out of the reach of local amputees.

Methods

The fact that 3D printing is a very recent technological development means that there is not much available literature on the subject. It is clear that this technology is in its infancy and therefore not much has been written on its form and its use. This fact is even more pronounced when it comes to existing data on the availability and application of 3D printing in developing countries such as Grenada. Despite this, we have made use of available data that can help in understanding why the introduction of 3D printing in countries like Grenada will widen and deepen the availability of adequate health care.

Data were collected from primary (local) sources such as the Diabetic Association and the Ministry of Health of Grenada and also from secondary sources such as the internet. These data were helpful in showing how much Grenada needs less costly delivery of body parts and therefore the dire need for the introduction of 3D printing. In several discussions with medical practitioners and other persons involved in the health care industry, it became clear that while 3D printing is not yet part of the health care delivery sector in Grenada its introduction will no doubt be welcomed. Its success can only be achieved through the partnership of IT and medical professionals.

Results

The fundamental issue facing us is how will the new technology impact on health care delivery in Grenada? Can we and will we benefit from the development of this technology? Diabetes and hypertension are the two most common diseases in Grenada affecting both our youth and our elderly.

While our health care providers have been focused on trying to prevent the continued increase of diseases such as hypertension and diabetes they must also focus on treatment of the hundreds of patients that are already affected. The graph in Figure 1 shows that hypertension cases in Grenada almost doubles that of diabetes cases. Both hypertension and diabetes result in

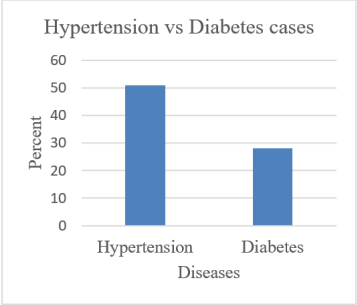

Figure 1: Hypertension and diabetes cases in Grenada

serious kidney disease and the sometimes inevitable amputations of limbs. These conditions have been exacerbated as a result of the high cost of care which includes expensive medication and dialysis. With the steady rise in the incidence of these non-communicable diseases, the inevitable result is that many of the patients require very expensive and most times unavailable kidney transplants and artificial limbs.

Data collected on lower limb amputations shown in Figure 2 seems to suggest an average of 45.2 amputations per year with a high on the odd years and a low on the even years for the given period. However, more data will have to be collected to ascertain this trend.

The production of body parts is by far the area of health delivery that will benefit the most from the development of 3D printing. According to Scott Dunham (2015), senior analyst for SmarTech Publishing, "one of the most inspiring outcomes from the rapid expansion of 3D printed medical solutions is the ability to help patients who previously had little hope for treatment." Dunham further stated that millions of persons across the world are currently unable to access critical surgeries because of financial difficulties or the uniqueness of their medical condition. He argued that with the advent of 3D printing this will change. Traditional transradial prosthetics are extremely expensive costing up to $20,000.00 per device with additional problems such as degradation and the patient outgrowing

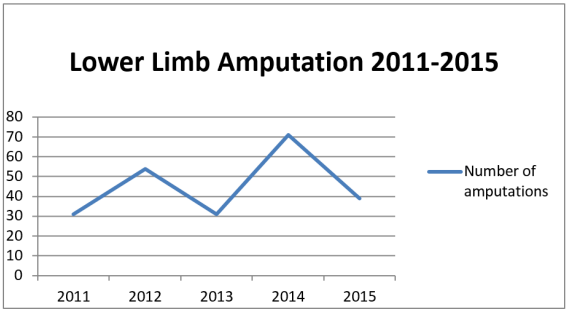

Figure 2: the number of limb amputation 2011 - 2015

the device. Many persons who need the devices simply cannot afford them.

Apart from limbs, 3D printing has also enabled mass production of other body parts, significant among these is the production of synthetic eyes on a mass scale. According to Fripp Design and Research, a UK based company, batch printing of up to 150 prosthetic eyes an hour has become a reality. The report states that this will result not only in lowering the cost of the prosthetic eyes but also the printing of each eye with slight variation in color.

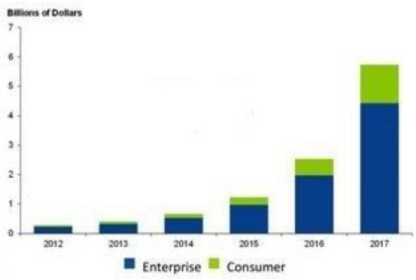

Figure 3: 3D printing sales (Source: Gartner, Forecast: 3D printers, worldwide, 2013

The graph in Figure 3 depicts a prediction by Gartner that 3D printing sales and the installed base will grow worldwide at a combined compound

annual growth rate of 59% through 2017, with the value of shipments attaining a 27% CAGR (Compound Annual Growth Rate) in the forecast period.

Discussion

There are several aspects of the Health Care Industry that will be impacted by 3D printing including production of body parts, the manufacturing of drugs, and the diagnosis of illnesses. While traditional methods will continue to be used in several areas as 3D technology develops, and becomes more accessible, we believe that it will become the method of choice for most health care providers. While the cost of the technology today may be prohibitive, it is expected that as is the case with most innovations it will eventually become less expensive and therefore more accessible even for small countries like Grenada.

The development of 3D printing raises the possibility not only of the availability of artificial limbs and replacement kidneys, but also the lowering of the cost of such care. A 3D kidney can be produced and delivered quickly to the recipient. Similarly, with the increased amputee rate in Grenada resulting from diabetes and other causes, this new technology will also make available at reasonable cost limbs for prosthetic purposes.

Apart from the availability of kidneys and limbs, the lowering of the cost to the consumer is also a critical factor in the case of Grenada. As a small developing nation with low incomes, transplants and prostheses are very much out of the financial reach of the average citizen. With 3D printing and the lowering of the cost, access to transplants and prosthetic limbs will be more widespread. This in our view is one of the main benefits to Grenadian citizens.

The fact that the cost of development and implementation in the early stages will be expensive means that the technology will take some time before it becomes readily available to the average citizen. While this technology has been available since the 1980's in the major industrialized countries we have not seen its use in the poorer less developed countries like Grenada. At the same time, Grenada must push for early access through contacts with the advanced countries in the form of development assistance and transfer of technologies to the less developed countries. It may be that the access to the new technology with its potential impact on large populations should be sought as a regional initiative. Rather than Grenada or any individual country developing the technology, we should seek to initially access 3D printing as a region through a regional body.

Conclusions

We have found that 3D printing is a fascinating new technological advancement that will change the world as we know it, particularly in the area of health care. This research shows that 3D printing can be used for producing body parts such as prosthetic legs and hands which will be more affordable for Grenadian amputees. Prosthetic limbs made by some other method are very expensive and out of the reach for many amputees. This paper shows a relatively high rate of amputees in Grenada. For those who have undergone limb amputation in Grenada a prosthetic limb can make their lives more comforting. 3D printing is enabling doctors and other health care professionals to treat more patients, without sacrificing results. However, this cannot be done without the partnership of professionals from both IT and medicine.

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Sequential Algorithm LU-decomposition of matrices over idempotent semifields

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Abstract

The present report is devoted to the problem of LU-decomposition of matrices over idempotent semifields. This problem can be solved in different ways. We consider the sequential algorithm.

Let R – idempotent semifield, LU-decomposition of the matrix $A \in R^{n\times n}$ are called lower triangular matrix $L \in R^{n\times n}$, and upper triangular matrix $U \in R^{n\times n}$, for which $A = LU$.

Theorem 1. Let $A \in R^{n\times n},\ A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},\ a_{11} \neq \mathbb{0}.$
If at least one of the inequalities $\frac{a_{i1}a_{1j}}{a_{11}} \leq_T a_{ij},\ i = 2,\dots,n,\ j = 2,\dots,n,$ is not satisfied, then the LU-decomposition the matrix A does not exist.

Proof. Let

$$L = \begin{pmatrix} l_{11} & \mathbb{0} & \dots & \mathbb{0} \\ l_{21} & l_{22} & \dots & \mathbb{0} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{pmatrix},\ U = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ \mathbb{0} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{0} & \mathbb{0} & \dots & u_{nn} \end{pmatrix}.$$

Then $LU = \begin{pmatrix} l_{11}u_{11} & l_{11}u_{12} & \dots & l_{11}u_{1n} \\ l_{21}u_{11} & l_{21}u_{12} \oplus l_{22}u_{22} & \dots & l_{21}u_{1n} \oplus l_{22}u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1}u_{11} & l_{n1}u_{12} \oplus l_{n2}u_{22} & \dots & l_{n1}u_{1n} \oplus \dots \oplus l_{nn}u_{nn} \end{pmatrix}.$

Of $l_{11}u_{11} = a_{11}$ it follows that $l_{11} = \frac{a_{11}}{u_{11}}$ for any $u_{11} \neq \mathbb{0}$. Of $l_{11}u_{12} = a_{12}$ we get $u_{12} = \frac{a_{12}}{l_{11}} = \frac{a_{12}u_{11}}{a_{11}}$. Similarly, we find u_{13}, and so on, $u_{1n} = \frac{a_{1n}u_{11}}{a_{11}}$. Of $l_{21}u_{11} = a_{21}$ it follows $l_{21} = \frac{a_{21}}{u_{11}}$. And so on, $l_{n1} = \frac{a_{n1}}{u_{11}}$.

Because the $l_{21}u_{12} \oplus l_{22}u_{22} = a_{22}$, then $l_{21}u_{12} \leq_T a_{22}$. Otherwise the equality will not be executed. $l_{21}u_{12} = \frac{a_{21}}{u_{11}}\frac{a_{12}u_{11}}{a_{11}} = \frac{a_{21}a_{12}}{a_{11}} \leq_T a_{22}$. Because the $l_{21}u_{13} \oplus l_{22}u_{23} = a_{23}$, then $l_{21}u_{13} = \frac{a_{21}a_{13}}{a_{11}} \leq_T a_{23}$, and so on. Similarly, we get: $l_{21}u_{1n} = \frac{a_{21}a_{1n}}{a_{11}} \leq_T a_{2n},\ l_{n1}u_{12} = \frac{a_{n1}a_{12}}{a_{11}} \leq_T a_{n2},$
$l_{n1}u_{1n} = \frac{a_{n1}a_{1n}}{a_{11}} \leq_T a_{nn}$.

Thus, if at least one of the inequalities $\frac{a_{i1}a_{1j}}{a_{11}} \leq_T a_{ij},\ i = 2, \ldots, n,$ $j = 2, \ldots, n$, is not satisfied, then LU-decomposition does not exist for matrix A.

We introduce the following notation: operation $\oplus' = \begin{cases} min, \oplus = max; \\ max, \oplus = min. \end{cases}$

Theorem 2. There is an algorithm to determine the impossibility or search LU-decomposition given matrix over an idempotent semifield, having complexity $O(n^3)$.

Proof. Let $A \in R^{n\times n},\ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{pmatrix},$

$a_{11} \neq \mathbb{0}$.

We must find the matrix:

$$L = \begin{pmatrix} l_{11} & \mathbb{0} & \mathbb{0} & \ldots & \mathbb{0} \\ l_{21} & l_{22} & \mathbb{0} & \ldots & \mathbb{0} \\ l_{31} & l_{32} & l_{33} & \ldots & \mathbb{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \ldots & l_{nn} \end{pmatrix},\ U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \ldots & u_{1n} \\ \mathbb{0} & u_{22} & u_{23} & \ldots & u_{2n} \\ \mathbb{0} & \mathbb{0} & u_{33} & \ldots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \ldots & u_{nn} \end{pmatrix}.$$

The product of these matrices over idempotent semifield R should coincide with the matrix A.

$$LU = \begin{pmatrix} l_{11}u_{11} & l_{11}u_{12} & \ldots & l_{11}u_{1n} \\ l_{21}u_{11} & l_{21}u_{12} \oplus l_{22}u_{22} & \ldots & l_{21}u_{1n} \oplus l_{22}u_{2n} \\ l_{31}u_{11} & l_{31}u_{12} \oplus l_{32}u_{22} & \ldots & l_{31}u_{1n} \oplus l_{32}u_{2n} \oplus l_{33}u_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1}u_{11} & l_{n1}u_{12} \oplus l_{n2}u_{22} & \ldots & l_{n1}u_{1n} \oplus \ldots \oplus l_{nn}u_{nn} \end{pmatrix}.$$

We will build decomposition in stages, first the first column of the matrix L and the first row of the matrix U, then the second column of the matrix L and the second row of the matrix U, and so on.

Let $\frac{a_{i1}a_{1j}}{a_{11}} \leq_T a_{ij},\ i = 2, \ldots, n,\ j = 2, \ldots, n$. Otherwise, by Theorem 1 LU-decomposition does not exist. Then we can build the first column of the matrix L and the first row of the matrix U. Let $l_{11} = \mathbb{1}$, $u_{11} = a_{11}$. Consequently $l_{21} = \frac{a_{21}}{a_{11}}, \ldots,\ l_{n1} = \frac{a_{n1}}{a_{11}}$, $u_{12} = a_{12}, \ldots,\ u_{1n} = a_{1n}$.

We will build the second column of the matrix L and the second row of the matrix U. We need to ensure there is no conflict at all the second terms in the matrix LU.

Consider the case of $a_{22} \neq \mathbb{0}$. Let $l_{22} = \mathbb{1},\ u_{22} = a_{22}$, $u_{23} = a_{23}, \ldots,\ u_{2n} = a_{2n}$.

If $l_{31}u_{12} = \frac{a_{31}a_{12}}{a_{11}} = a_{32}$, then $l_{32} \leq_T \frac{a_{32}}{u_{22}},\ l_{32} \leq_T \frac{a_{32}}{a_{22}}$. In addition: $l_{32}u_{23} \leq_T a_{33},\ l_{32}u_{24} \leq_T a_{34}, \ldots,\ l_{32}u_{2n} \leq_T a_{3n}$. Consequently, $l_{32} \leq_T \frac{a_{33}}{a_{23}},\ l_{32} \leq_T \frac{a_{34}}{a_{24}}, \ldots,\ l_{32} \leq_T \frac{a_{3n}}{a_{2n}}$. Thus,

$$l_{32} = \frac{a_{32}}{a_{22}} \oplus' \frac{a_{33}}{a_{23}} \oplus' \frac{a_{34}}{a_{24}} \oplus' \ldots \oplus' \frac{a_{3n}}{a_{2n}}.$$

If any number in the denominator $a_{2p} = \mathbb{0},\ p = 3, \ldots, n$, then $u_{2p} = \mathbb{0}$. Then $l_{32}u_{2p} = \mathbb{0}$, therefore, this term does not affect the formation of a_{3p}, and $\frac{a_{3p}}{a_{2p}}$ we can rule out.

If $l_{31}u_{12} = \frac{a_{31}a_{12}}{a_{11}} <_T a_{32}$, then $l_{32} = \frac{a_{32}}{u_{22}},\ l_{32} = \frac{a_{32}}{a_{22}}$. If $\frac{a_{32}}{a_{22}} \leq_T \frac{a_{33}}{a_{23}} \oplus' \frac{a_{34}}{a_{24}} \oplus' \ldots \oplus' \frac{a_{3n}}{a_{2n}}$, there is no contradiction. Similarly we are looking for $l_{42}, \ldots,\ l_{n2}$. Thus, we find the second column of the matrix L and the second row of the matrix U. If $\frac{a_{32}}{a_{22}} >_T \frac{a_{33}}{a_{23}} \oplus' \frac{a_{34}}{a_{24}} \oplus' \ldots \oplus' \frac{a_{3n}}{a_{2n}}$ or a similar relationship obtained by the calculation $l_{42}, \ldots,\ l_{n2}$, then the there are contradictions. In this case, we will build the second row of the matrix U, with values $l_{32} = \frac{a_{32}}{a_{22}}, \ldots,\ l_{n2} = \frac{a_{n2}}{a_{22}}$.

If $l_{21}u_{13} = \frac{a_{21}a_{13}}{a_{11}} = a_{23}$, then $u_{23} \leq_T \frac{a_{23}}{l_{22}},\ u_{23} \leq_T a_{23}$. In addition: $l_{32}u_{23} \leq_T a_{33},\ l_{42}u_{23} \leq_T a_{43}, \ldots,\ l_{n2}u_{23} \leq_T a_{n3}$. Consequently, $u_{23} \leq_T \frac{a_{33}a_{22}}{a_{32}},\ u_{23} \leq_T \frac{a_{43}a_{22}}{a_{42}}, \ldots,\ u_{23} \leq_T \frac{a_{n3}a_{22}}{a_{n2}}$. Thus,

$$u_{23} = a_{23} \oplus' \frac{a_{33}a_{22}}{a_{32}} \oplus' \frac{a_{43}a_{22}}{a_{42}} \oplus' \ldots \oplus' \frac{a_{n3}a_{22}}{a_{n2}}.$$

The terms in which the denominator is $\mathbb{0}$, we exclude.

If $l_{21}u_{13} = \frac{a_{21}a_{13}}{a_{11}} <_T a_{23}$, then $u_{23} = \frac{a_{23}}{l_{22}},\ u_{23} = a_{23}$. If $a_{23} \leq_T \frac{a_{33}a_{22}}{a_{32}} \oplus' \frac{a_{43}a_{22}}{a_{42}} \oplus' \ldots \oplus' \frac{a_{n3}a_{22}}{a_{n2}}$, there is no contradiction. Similarly, we find $u_{24}, \ldots,\ u_{2n}$. Thus, we found a second row matrix U and the second column of the matrix L. If $a_{23} >_T \frac{a_{33}a_{22}}{a_{32}} \oplus' \frac{a_{43}a_{22}}{a_{42}} \oplus' \ldots \oplus' \frac{a_{n3}a_{22}}{a_{n2}}$ or a similar relationship obtained by the calculation $u_{24}, \ldots,\ u_{2n}$, then the there are contradictions. Consequently, LU-decomposition does not exist.

Consider the case of $a_{22} = \mathbb{0}$. Of $l_{21}u_{12} \oplus l_{22}u_{22} = \mathbb{0}$ it follows that $l_{22}u_{22} = \mathbb{0}$. If $u_{22} = \mathbb{0}$, then $l_{31}u_{12} = a_{32},\ l_{41}u_{12} = a_{42}, \ldots,\ l_{n1}u_{12} = a_{n2}$. Let $l_{22} = \mathbb{1}$, then $u_{23} = a_{23}, \ldots,\ u_{2n} = a_{2n}$. Then

$$l_{i2} = \frac{a_{i3}}{a_{23}} \oplus' \frac{a_{i4}}{a_{24}} \oplus' \ldots \oplus' \frac{a_{in}}{a_{2n}},$$

where $i = 3, \ldots, n$. If $l_{22} = \mathbb{0}$, then $l_{21}u_{13} = a_{23}, \ldots,\ l_{21}u_{1n} = a_{2n}$. Let $u_{22} = \mathbb{1}$, then $l_{32} = a_{32}, \ldots,\ l_{n2} = a_{n2}$. Then

$$u_{2j} = \frac{a_{3j}}{a_{32}} \oplus' \frac{a_{4j}}{a_{42}} \oplus' \ldots \oplus' \frac{a_{nj}}{a_{n2}},$$

where $j = 3, \ldots, n$. The terms in which the denominator is $\mathbb{0}$, we exclude. If all terms are excluded, is set to $\mathbb{0}$.

Then in the same way we consider the third terms in the matrix of the LU and build a third column of the matrix L and the third row of the matrix U. Case $a_{33} = \mathbb{0}$, we consider similar to the case $a_{22} = \mathbb{0}$. And so on up to $n-1$. $l_{nn} = \mathbb{1},\ u_{nn} = a_{nn}$.

Consider the case of $a_{11} = \mathbb{0}$. It means that $l_{11}u_{11} = \mathbb{0}$. If $l_{11} = \mathbb{0}$, then the first row of the matrix A should be zero. Let $u_{11} = \mathbb{1}$, then $l_{21} = a_{21}, \ldots,\ l_{n1} = a_{n1}$. We find u_{1j} in the following way:

$$u_{1j} = \frac{a_{2j}}{a_{21}} \oplus' \frac{a_{3j}}{a_{31}} \oplus' \frac{a_{4j}}{a_{41}} \oplus' \ldots \oplus' \frac{a_{nj}}{a_{n1}},$$

where $j = 2, \ldots, n$. If $u_{11} = \mathbb{0}$, then the first column of the matrix A must be zero. Let $l_{11} = \mathbb{1}$, then $u_{12} = a_{12}, \ldots,\ u_{1n} = a_{1n}$. We find

$$l_{i1} = \frac{a_{i2}}{a_{12}} \oplus' \frac{a_{i3}}{a_{13}} \oplus' \frac{a_{i4}}{a_{14}} \oplus' \ldots \oplus' \frac{a_{in}}{a_{1n}},$$

where $i = 2, \ldots, n$. The terms in which the denominator is $\mathbb{0}$, we exclude. If all terms are excluded, is set to $\mathbb{0}$. Thus, at $a_{11} = \mathbb{0}$, if the first row or the first column of the matrix A not equal to $\mathbb{0}$, then LU-decomposition does not exist.

We fill the matrix $L,\ U$ initial values:

$$L = \begin{pmatrix} l_{11} & \mathbb{0} & \mathbb{0} & \ldots & \mathbb{0} \\ l_{21} & l_{22} & \mathbb{0} & \ldots & \mathbb{0} \\ l_{31} & l_{32} & l_{33} & \ldots & \mathbb{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \ldots & l_{nn} \end{pmatrix},\ U = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ \mathbb{0} & a_{22} & a_{23} & \ldots & a_{2n} \\ \mathbb{0} & \mathbb{0} & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \ldots & a_{nn} \end{pmatrix},$$

where

$$l_{ij} = \begin{cases} a_{ij}a_{jj}^{-1}, \text{ if } a_{jj} \neq \mathbb{0}; \\ a_{ij}, \text{ if } a_{jj} = \mathbb{0}. \end{cases}$$

In the algorithm, the values of the matrix elements L and U will be changed, if necessary.

Algorithm.

Input: $A \in R^{n \times n}$.
Output: null, if LU-decomposition does not exist, or pare matrices: $L,\ U \in R^{n \times n}$.
Set the initial values of the elements in the matrices $L,\ U$;
if $a_{11} = \mathbb{0}$ {
  if $a_{1j} = \mathbb{0}$ {\\$j = 2, \ldots, n$
    $u_{11} = \mathbb{1}$;
    for $j = 2$ **to** n **do**{ $u_{1j} = \infty$;
      for $k = 2$ **to** n **do**{ **if** $a_{k1} \neq \mathbb{0}$ { $u_{1j} = u_{1j} \oplus' \frac{a_{kj}}{a_{k1}}$; } }
      if $u_{1j} = \infty$ { $u_{1j} = \mathbb{0}$; } }
  }**else** { **if** $a_{i1} = \mathbb{0}$ {\\$i = 2, \ldots, n$

$l_{11} = \mathbb{1}$;
for $i = 2$ **to** n **do**{ $l_{i1} = \infty$;
for $k = 2$ **to** n **do**{ **if** $a_{1k} \neq \mathbb{0}$ { $l_{i1} = l_{i1} \oplus' \frac{a_{ik}}{a_{1k}}$; } }
if $l_{i1} = \infty$ { $l_{i1} = \mathbb{0}$; } } }**else** { **return** *null*; } }
}**else** { **for** $i = 2$ **to** n **do**{
for $j = 2$ **to** n **do**{ **if** $\frac{a_{i1}a_{1j}}{a_{11}} >_T a_{ij}$ { **return** *null*; } } } }
for $m = 2$ **to** $n - 1$ **do**{ $f = false$; $h = 0$;
for $t = m + 1$ **to** n **do**{ $v_t = l_{tm}$; }
for $i = m + 1$ **to** n **do**{ $s = \infty$;
for $p = m$ **to** n **do**{ **if** $a_{mp} \neq \mathbb{0}$ { $s = s \oplus' \frac{a_{ip}}{a_{mp}}$; } }
if $s = \infty$ { $s = \mathbb{0}$; }
$flag = false$;
for $k = 1$ **to** $m - 1$ **do**{ **if** $l_{ik}u_{km} = a_{im}$ { $flag = true$; *end for*; } }
if $flag = true$ { $v_i = s$; $h = h + 1$;
}**else** { **if** $a_{mm} \neq \mathbb{0}$ { **if** $s \neq \frac{a_{im}}{a_{mm}}$ { $f = true$; *end for*; } }**else** { *end for*; } } }
if $a_{mm} = \mathbb{0}$ { **if** $h = n - m$ { $l_{mm} = \mathbb{1}$;}**else**{ $f = true$; } }
if $f = false$ { **for** $q = m + 1$ **to** n **do**{ $l_{qm} = v_q$; }
}**else**{ $h = 0$;
for $j = m + 1$ **to** n **do**{
if $a_{mm} \neq \mathbb{0}$ { $s = a_{mj}$;
for $p = m + 1$ **to** n **do**{ **if** $a_{pm} \neq \mathbb{0}$ { $s = s \oplus' \frac{a_{pj}a_{mm}}{a_{pm}}$; } }
}**else**{ $s = \infty$;
for $p = m + 1$ **to** n **do**{ **if** $a_{pm} \neq \mathbb{0}$ { $s = s \oplus' \frac{a_{pj}}{a_{pm}}$; } }
if $s = \infty$ { $s = \mathbb{0}$; } }
$flag = false$;
for $k = 1$ **to** $m - 1$ **do**{ **if** $l_{mk}u_{kj} = a_{mj}$ { $flag = true$; *end for*; } }
if $flag = true$ { $u_{mj} = s$; $h = h + 1$;
}**else** { **if** $a_{mm} \neq \mathbb{0}$ { **if** $s \neq a_{mj}$ { **return** *null*; } }**else** { *end for*; } } }
if $a_{mm} = \mathbb{0}$ { **if** $h = n - m$ { $u_{mm} = \mathbb{1}$; }**else**{ **return** *null*; } } } }
return L, U;

The complexity of the algorithm — $O(n^3)$.

Acknowledgments

The work is partially supported by RFBR grant No 16-07-00420.

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About Probabilistic Algorithm of Exact Matrix Inversion

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Abstract

We discuss the problem of constructing an effective algorithm for the integer matrices inversing. We wish to construct an agorithm with complexity about $\tilde{n}^3$ bit operations. We discuss one of the way which is based on the Smith form.

Introduction

We discuss the problem of constructing an effective algorithm for the integer matrix inversing. Best algorithm was proposed by Arne Storiohanom [1]. The algorithm has complexity $\tilde{n}^3(log(||A|| + log||A^{-1}||))$ He proposed a probabilistic algorithm that with probability at least $1/2$ computes the inverse matrix for the non-singular integral matrix A size $n\ timesn$ and uses

$$\sim n^3(log||A|| + log||A^{-1}||)$$

bit operations. Here $||A|| = \max_{i,j} |A_{ij}|$, symbol sim is the missing factor

$$a\ log(n)^b(log\ log(||A||))^c,$$

and the numbers a, b, c – is some positive constant.

Best known before that algorithm had a bit complexity

$$\sim n^{w+1}log||A||.$$

Random matrix with a high probability is well-conditioned and has $||A|| \approx ||A^{-1}||$. However, it may be that $||A^{-1}|| \approx n||A||$ for ill-conditioned matrix, for example, when $\det(A) = 1$. Thus, Storiohana algorithm allows us to calculate the inverse matrix faster for well-conditioned matrix ($\sim n^3log||A||$). And this algorithm does not improve in the case of ill-conditioned matrix ($\sim n^4log||A||$).

The central idea of this algorithm is to calculate the Smith form of the initial matrix as a sum of matrices of rank one.

Let $\mathbf{r} = \mathrm{rank}(A)$

$$\mathbf{snf}(A) = S = PAQ = Diag(s_1, s_2, ..., s_{\mathbf{r}}, 0, 0, ..., 0)$$

– is the Smith form of matrix A, P and Q – are unimodular matrices and $\forall_i\ s_i | s_{i+1}$. Then the expansion of Smith form can be written as follows

$$A = (s_1)c_1r_1 + (s_2)c_2r_2 + ... + (s_{\mathbf{r}})c_{\mathbf{r}}r_{\mathbf{r}}. \quad (1)$$

here $c_i r_i,\ i = 1, 2, .., r$ is an outer product of the column c_i by row r_i.

The existence of such an expansion follows directly from the following matrix identity.

Let $s_1 = gcd(A)$, w_1 and h_1 – is a row and column, satisfying the equation $w_1 A h_1 = s_1$ and let $r_1 = w_1 A / s_1$, $c_1 = A h_1 / s_1$. Then we have the following matrix identity:

$$\begin{bmatrix} 1 & w_1 \\ -c_1 & I_n - c_1 w_1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} 1 & -r_1 \\ h_1 & I_n - h_1 r_1 \end{bmatrix} = \begin{bmatrix} s_1 & 0 \\ 0 & A - s_1 c_1 r_1 \end{bmatrix}. \quad (2)$$

On the left side of equality both factors are the unimodular matrices. Hence the matrix A and $\mathbf{diag}(s_1, A - s_1 c_1 r_1)$ have the same Smith form. And we can easy to find them using this recursive identity: first, the matrix A, then the matrix $A_2 = A - s_1 c_1 r_1$, and so on.

As was shown in [1] for $\mathbf{r} = n$ the algorithm requires a $\sim n^3(log||A||)$ bit operations. We must each time taking random vector h_i, then the vector w_i we must find using the extended Euclidean algorithm, calculating $gcd(A_i h_i)$. The equality $gcd(A_i h_i) = gcd(A_i)$ will be true with very high probability.

The following is an algorithm for computing the inverse matrix, which has roughly the same complexity in operations on integer coefficients. Its bit complexity we have to evaluate in the future.

Computation of the inverse matrix

Suppose we have already constructed the decomposition of Smith form of the matrix A and calculated all the components s_i, w_i, h_i, r_i, c_i $(i = 1, 2, .., \mathbf{r})$ in decomposition (1).

We will introduce other notations. We further denote by w_i, r_i, h_i, c_i matrix of size $n \times n$, all of whose elements are zero except for one i-th row, which is equal to w_i or r_i, or one i th column, which equals h_i or c_i, respectively. In the new notation Smith decomposition will be recorded in the same manner as in (1).

THEOREM.

Let $A = (s_1)c_1r_1 + (s_2)c_2r_2 + ... + (s_{\mathbf{r}})c_{\mathbf{r}}r_{\mathbf{r}}$ – be Smith decomposition for matrix A, of size $n \times n$, $\mathbf{r} = \mathrm{rank}(A)$, $r_1 = w_1A/s_1$, $c_1 = Ah_1/s_1$. Let S'_i – be $n \times n$ matrix, which is different from zero only one diagonal element in the i th row, which is equal s_i, and let $S_i = S'_1 + S'_2 + .. + S'_i$ $(1 \le i \le \mathbf{r})$ and $S_0 = 0$.

Let $F_i = I_n - c_iw_i$, $G_i = I_n - h_ir_i$, $A_i = (s_i)c_ir_i + ... + (s_{\mathbf{r}})c_{\mathbf{r}}r_{\mathbf{r}}$,

$$U_i = \begin{bmatrix} I_n & w_i \\ -c_i & F_i \end{bmatrix} \text{ Đž } V_i = \begin{bmatrix} I_n & -r_i \\ h_i & G_i \end{bmatrix} \quad (i = 1, .., \mathbf{r}.)$$

Then the following matrix identities hold for any integer k, 1 *lek le* $bfr(A)$:k, $1 \le k \le \mathbf{r}(A)$:

$$\begin{bmatrix} I_n & w_k \\ -c_k & F_k \end{bmatrix} \begin{bmatrix} S_{k-1} & 0 \\ 0 & A_k \end{bmatrix} \begin{bmatrix} I_n & -r_k \\ h_k & G_k \end{bmatrix} = \begin{bmatrix} S_k & 0 \\ 0 & A_{k+1} \end{bmatrix}, \quad (3)$$

$$U_kU_{k-1} \cdots U_1 = \begin{bmatrix} \mathbf{L}_k & \mathbf{W}_k \\ -\mathbf{C}_k & \mathbf{F}_k \end{bmatrix}, \; V_1V_2 \cdots V_k = \begin{bmatrix} \mathbf{M}_k & -\mathbf{R}_k \\ \mathbf{H}_k & \mathbf{G}_k \end{bmatrix}, \quad (4)$$

$$\mathbf{W}_{\mathbf{r}}A\mathbf{H}_{\mathbf{r}} = S_{\mathbf{r}}, \quad (5)$$

in which the the notation used

$$\mathbf{F}_k = F_kF_{k-1} \cdots F_1, \quad \mathbf{G}_k = G_1G_2 \cdots G_k$$

$\mathbf{W}_k = w_1 + w_2\mathbf{F}_1 + .. + w_k\mathbf{F}_{k-1}$, $\mathbf{C}_k = c_k + F_{k-1}(c_{k-1} + F_{k-2}(c_{k-2} + .. + F_1(c_1)..)$, $\mathbf{H}_k = h_1 + \mathbf{G}_1h_2 + .. + \mathbf{G}_{k-1}h_k$, $\mathbf{R}_k = (..(r_1)G_1 + .. + r_{k-2}G_{k-2}) + r_{k-1})G_{k-1} + r_k$, $\mathbf{L}_k = \mathbf{I}_n - (w_2\mathbf{C}_1 + w_3\mathbf{C}_2 + .. + w_k\mathbf{C}_{k-1})$, $\mathbf{M}_k = \mathbf{I}_n - (\mathbf{R}_1h_2 + \mathbf{R}_2h_3 + .. + \mathbf{R}_{k-1}h_k)$

PROOF:

The identity (2), which is used in the first step of calculating Smith decomposition, can be written in the form in which it will look for the step i. At the same time, we extend it zero and unit elements. And besides, we will add a diagonal matrix S_{i-1} and $S_i = S_{i-1} + S'_i$ to the left and the right side. Here, obviously, the identity is retained since $c_iS_{i-1} = 0$. As a result, come to identity (3). Â We prove (4) by induction. For k = 1 the assertion is obvious. Suppose it is true for some $k \ge 1$. Let us prove the following equality

$$\begin{bmatrix} I_n & w_{k+1} \\ -c_{k+1} & F_{k+1} \end{bmatrix} \begin{bmatrix} \mathbf{L}_k & \mathbf{W}_k \\ -\mathbf{C}_k & \mathbf{F}_k \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{k+1} & \mathbf{W}_{k+1} \\ -\mathbf{C}_{k+1} & \mathbf{F}_{k+1} \end{bmatrix}$$

Matrices $\mathbf{L}_k$ and $\mathbf{W}_k$ differ from the unit and zero matrices, respectively, only in the first k rows therefore $c_{k+1}\mathbf{L}_k = c_{k+1}$ Ðž $c_{k+1}\mathbf{W}_k = 0$. This implies that: $\mathbf{F}_{k+1} = F_{k+1}\mathbf{F}_k$, $\mathbf{C}_{k+1} = c_{k+1} + F_{k+1}\mathbf{C}_k$, $\mathbf{W}_{k+1} = \mathbf{W}_k + w_{k+1}\mathbf{F}_k, \mathbf{L}_{k+1} = \mathbf{L}_k - w_{k+1}\mathbf{C}_k$.

The second of the identities (4) can be proved similarly.

Â To prove the identity (5) applies to the original matrix A k times the equation (3) and use (4). Â A result we get

$$\begin{bmatrix} \mathbf{L}_k & \mathbf{W}_k \\ -\mathbf{C}_k & \mathbf{F}_k \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & A_k \end{bmatrix} \begin{bmatrix} \mathbf{M}_k & -\mathbf{R}_k \\ \mathbf{H}_k & \mathbf{G}_k \end{bmatrix} = \begin{bmatrix} S_k & 0 \\ 0 & A_{k+1} \end{bmatrix}$$

When $k = bfr$ from this equation, we obtain (5).

THEOREM

Let all the conditions of Theorem 1 are satisfied, and rank$A = n$. Then there is a factorization for the the inverse matrix:

$$A^{-1} = \mathbf{H}_n S^{-1} \mathbf{W}_n. \tag{6}$$

Here

$$\mathbf{H}_k = h_1 + \mathbf{G}_1 h_2 + .. + \mathbf{G}_{k-1} h_k,\ \mathbf{G}_k = G_1 G_2 \cdots G_k,\ G_i = I_n - h_i r_i, \tag{7}$$

$$\mathbf{W}_k = w_1 + w_2 \mathbf{F}_1 + .. + w_k \mathbf{F}_{k-1},\ \mathbf{F}_k = F_k F_{k-1} \cdots F_1,\ F_i = I_n - c_i w_i. \tag{8}$$

The proof is reduced to the inversion of equality (5).

Complexity. Â Â Let us find a product

$$(I_n - h_1 r_1)(I_n - h_2 r_2) \cdots (I_n - h_{n-1} r_{n-1})$$

$$G_1 G_2 = (I_n - h_1 r_1)(I_n - h_2 r_2) = (I_n - h_1(r_1 - \mu_{12} r_2) - h_2 r_2)$$

$$G_1 G_2 G_3 = (I_n - h_1(r_1 - q_1 r_2) - h_2 r_2)(I_n - h_3 r_3) =$$

$$(I_n - h_1(r_1 - q_1 r_2) - h_2 r_2) - (h_3 r_3 - h_1(r_1 - q_1 r_2) h_3 r_3 - h_2 r_2 h_3 r_3) =$$

$$I_n - h_1(r_1 - \mu_{12} r_2 - (\mu_{13} - \mu_{12}\mu_{23}) r_3) - h_2(r_2 - \mu_{23} r_3) - h_3 r_3 =$$

$$I_n - (h_1 q_1 q_2 .. q_{n-1} + h_2 q_2 q_3 .. q_{n-1} - h_3 q_3 q_4 .. q_{n-1} ... + h_{n-1}) r_{n-1}$$

$$q_i = r_i h_{i+1}$$

–is the value of the element (i, i + 1) in the matrix (i = 1..n-2) Â To calculate the matrix $\mathbf{H}_n$ need to calculate each of its columns in accordance with (7). Column number k is equal to

$$G_1 G_2 \cdots G_{k-1} h_k.$$

We calculate it from right to left. We calculate the last product:

$$G_{k-1} h_k = (I_n - h_{k-1} r_{k-1}) h_k = h_k - h_{k-1}(r_{k-1} h_k)$$

To do this, the result of the scalar product of vectors $(r_{k-1} h_k)$ multiply by a column vector h_{k-1} and subtract from column h_{k-1}. In total, we performed 2n multiplications and additions as well. Continuing to go on like this, we calculate the entire column with the number k using $2(k-1)n$ operations. Since the number of columns is equal to n, it would take n^3 of operations for all calculations.

Similarly we can calculate the matrix $\mathbf{W}_n$.

Thus, if we know Smith decomposition of matrix A? then the inverse matrix factorization can be obtained for n^3 of operations over coefficients.

Acknowledgments

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[1] *A*rne Storjohann On the Complexity of Inverting Integer and Polynomial Matrices. Computational Complexity. 24 (2015), 777 â 821 DOI 10.1007/s00037-015-0106-7

About Effective Methods of Parallelizing Block Recursive Algorithms

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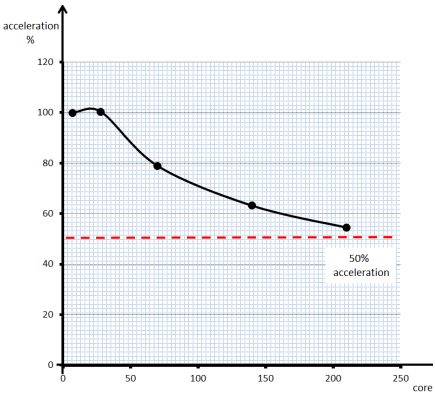
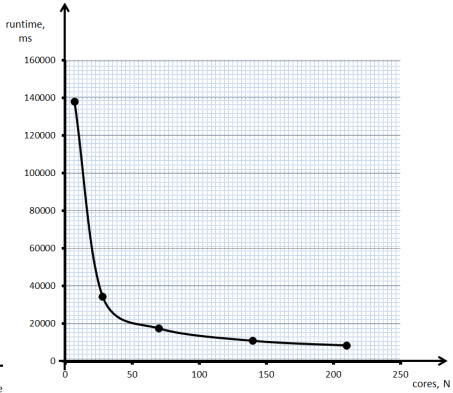
Abstract

We describe an algorihm for the decentralized control of parallel computing process which is based on the SPMD computational paradigm and present the results of experiments on a cluster "MVS-10P".



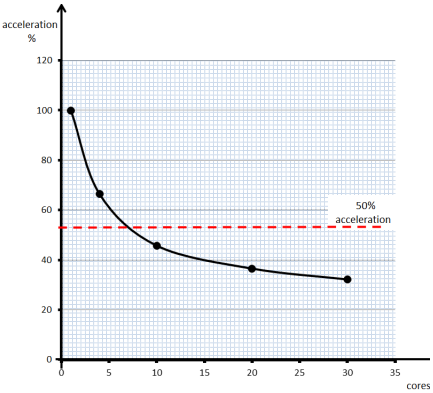
Im. 1. Acceleration calculate the product of two dense matrices 2048x2048 with increasing amounts of computing cluster cores.



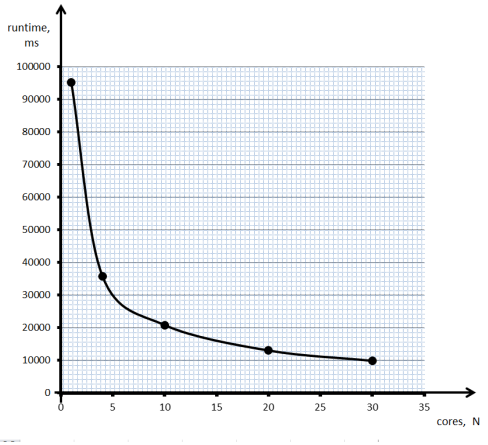
Im. 2. The time for calculating the product of two dense matrices 2048x2048 on the number of cluster computing cores.

Acknowledgments

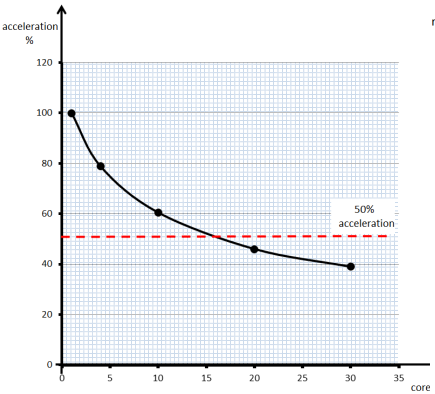
The work is partially supported by RFBR grant No 16-07-00420.



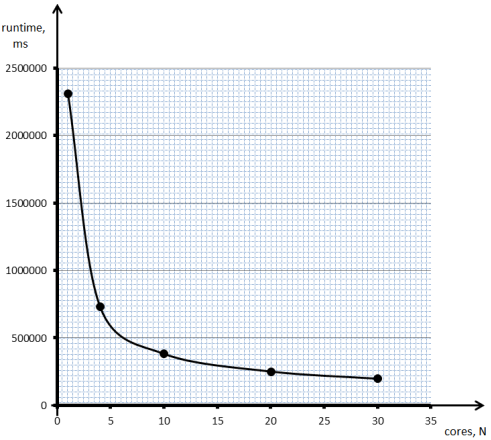
Im. 3. Acceleration calculate of the associated matrix with increasing amounts of computing cluster cores. The initial data is dense matrix size of 512x512.



Im. 4. The time for calculating the amount of the associated matrix computing cluster cores. The initial data is dense matrix size of 512x512.



Im. 5. Acceleration calculate of the associated matrix with increasing amounts of computing cluster cores. The initial data is dense matrix size of 1024x1024.



Im. 6. The time for calculating the amount of the associated matrix computing cluster cores. The initial data is dense matrix size of 1024x1024.

Permutation Model of Quantum Evolution: Search for Emergent Structures

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Abstract

The evolution of a quantum system is a sequence of observations (measurements) with unitary transitions between them. From a mathematical point of view, the measurement is a projection of a vector of a Hilbert space into the subspace specified by the measuring device. In conventional quantum mechanics it is assumed that between observations the system evolves deterministically according to the Schrödinger equation. In contrast to this simple deterministic picture, we assume that there is a bunch of all possible gauge connections (ways of identification of the state vectors in the transitions between adjacent observations), and the unique Schrödinger's evolution is only the dominant element in the bunch. We consider a constructive model based on the natural and standard representations of the symmetric group. Our strategy in the investigation of quantum-mechanical problems within this model consists in searching dominant structures and dominant evolutions. As the main computational tool we use the Monte Carlo method, which is inherently well suited to the "irreducible quantum randomness". The method demonstrates high efficiency and precision in our problems.

Introduction

Epistemic approach to explain the empirically observable randomness of quantum behavior implies the existence of "ontic" states (full information about them is unavailable) and "epistemic" states (which carry partial information about the quantum system and can be expressed in terms of vectors in a Hilbert space $\mathcal{H}$). Various, sometimes artificial, constructs are offered to explain the loss of information. For example, the Spekkens model is based on *the knowledge balance principle* [1]: "...for every system, at every time, the amount of knowledge one possesses about the ontic state of the system at that time must equal the amount of knowledge one lacks."

The "ontic" states in the model we consider here are collections of elements of a finite set Ω of size N, i.e. the ontic states are elements of the set $\mathcal{O} = \mathbb{N}^{\Omega}$, where $\mathbb{N} = \{0, 1, \ldots\}$ is the set of natural numbers. We assume that a group G acts by permutations on Ω with corresponding permutation representation in the module $\mathcal{O}$. The group G can be treated as a *gauge group*, since its elements define changes of the reference frame in the set Ω. The loss of information in the model occurs quite naturally: one can observe only *the relations that are invariant* under the action of G. More specifically, we will assume that the gauge group in our model is $\mathsf{G} = \mathrm{Sym}\,(\Omega) \simeq \mathsf{S}_{\mathsf{N}}$.

To construct a numerical system that takes into account the periodicity of elements of the group G, we need to add Pth roots of unity to the natural numbers. Here the period P depends on the structure of G. In particular, P divides the exponent of G. If $\mathrm{P} \geq 3$, then the fraction field of the linear combinations of natural numbers with Pth roots of unity, called the Pth *cyclotomic field*, is a dense subfield of the field $\mathbb{C}$ — the main field in quantum mechanics. In terms of the complex numbers, the "basic" primitive Pth root of unity can be written as $\mathrm{e}^{\frac{2\pi \mathrm{i}}{\mathrm{P}}}$. The cyclotomic field allows us to provide the set $\mathcal{O}$ with the structure of a Hilbert space $\mathcal{H}$. Now we can turn the permutation representation of G in the module $\mathcal{O}$ into the unitary representation $\mathsf{U}(\mathsf{G})$ in the space $\mathcal{H}$. As is known, the group S_{N} is a *rational-representation* group, i.e. any of its irreducible representations is realizable over the field $\mathbb{Q}$ or, equivalently, over the ring $\mathbb{Z}$. Thus, in the case $\mathsf{G} = \mathsf{S}_{\mathsf{N}}$, the period $\mathrm{P} = 2$ is sufficient, since $\mathbb{Z}$ is the extension of the semiring $\mathbb{N}$ by the 2nd primitive root of unity (which can be represented as (-1) or $\mathrm{e}^{\frac{2\pi \mathrm{i}}{2}}$).

To specify the model we use the N-dimensional *natural* and $(\mathsf{N}-1)$-dimensional irreducible *standard* representations of S_{N}. We consider the constructive version of natural representation, i.e. the above-mentioned permutation representation in the module $\mathcal{O}$. The constructive version of the standard representation is the projection of the natural representation into the $(\mathsf{N}-1)$-dimensional complement to the one-dimensional submodule of $\mathcal{O}$ consisting of elements of the form $(\underbrace{k, k, \ldots, k}_{\mathsf{N}})^T$, $k \in \mathbb{N}$.

Scheme of quantum evolution

The scheme shown in Figure 1 represents the quantum evolution with observations. Here $t_i \in \{0, 1, \ldots\}$ is *time of observation* (elementary ("Planck") time unit is 1); the *projector* $\Pi_{\psi_{t_i}} = |\psi_{t_i}\rangle\langle\psi_{t_i}|$ represents the measur-

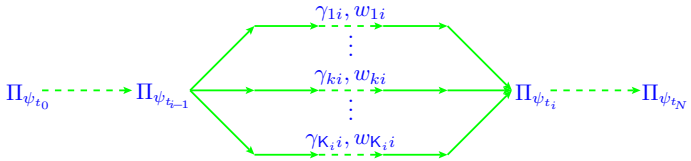

Figure 1: Quantum evolution with observations

ing device configured to the state ψ_{t_i}; γ_{ki} are all possible products of $\Delta t_i = t_i - t_{i-1}$ elements of a finite *gauge group* G with unitary representation U in $\mathcal{H}$; $w_{ki} \geq 0$ is the *weight* of γ_{ki}; K_i is the number of γ_{ki}'s.

Standard quantum mechanics implies a single unitary evolution between measurements, i.e. for some $a \in \mathsf{G}$ the product $a^{\Delta t_i}$ has weight 1, and all other products have zero weights. The unitary evolution can be written as $U = \mathsf{U}\left(a^{\Delta t_i}\right)$, or, introducing the *Hamiltonian* $H = \mathrm{i} \ln \mathsf{U}(a)$, as $U = \mathrm{e}^{-\mathrm{i} H \Delta t_i}$. However, we will study the case of general weights suggesting that a unique unitary evolution should occur as a dominant element in the set of all possible evolutions. The combinatorics of evolutions is rather complicated since the weights depend on both the properties of group elements and the state vectors. Besides, the behavior of the model — as is typical for quantum mechanics — depends on the choice of time intervals between measurements Δt_i.

Most likely trajectories and least action

The main problem in the study of the evolution is the search for the most probable trajectories. The *one-step transition probability* is given by the formula

$$\mathbb{P}_{\psi_{t_{i-1}} \to \psi_{t_i}} = \sum_{k=1}^{\mathsf{K}_i} w_{ki} \operatorname{tr}\left(\Pi_{\varphi_{ki}} \Pi_{\psi_{t_i}}\right), \quad \text{where} \quad \varphi_{ki} = \mathsf{U}(\gamma_{ki})\, \psi_{t_{i-1}}.$$

The probability of the whole trajectory is calculated as the product

$$\mathbb{P}_{\psi_{t_0} \to \cdots \to \psi_{t_N}} = \prod_{i=1}^{N} \mathbb{P}_{\psi_{t_{i-1}} \to \psi_{t_i}}. \tag{10}$$

It is convenient to replace the product by a sum. To do this, we introduce the *one-step entropy*

$$\mathbf{S}_{\psi_{t_{i-1}} \to \psi_{t_i}} = -\log \mathbb{P}_{\psi_{t_{i-1}} \to \psi_{t_i}}.$$

Continuum analog of the one-step entropy is the *Lagrangian* $\mathcal{L}$. Now the probability of trajectory (10) can be replaced by the *entropy of trajectory*

$$\mathbf{S}_{\psi_{t_0}\to\cdots\to\psi_{t_N}} = \sum_{i=1}^{N} \mathbf{S}_{\psi_{t_{i-1}}\to\psi_{t_i}}. \tag{11}$$

This is a discrete counterpart of the continuous *action* $\mathcal{S} = \int \mathcal{L} dt$. Search for the most probable trajectories, i.e. those that provide the *maximum* of expression (10), is equivalent to finding the *minimum* of expression (11). This is *the principle of least action.*

Quantum effects of gauge curvature

A single unitary evolution, being merely a coordinate transformation, is not adequate for describing the physical reality. One of the ways to obtain observable effects in the scenario of unitary evolution is a projective measurement. This is incorporated in the scheme shown in Figure 1, where we have a sequence of the "reference vectors", which are compared with the vectors obtained during the unitary evolutions. These reference vectors are known to us in advance, as they are defined by the settings of the measuring devices.

Description of the gauge effects requires a slightly different scheme. In the gauge theories, the observable effects are obtained by comparing the results of different evolutions. Metaphorically, we can treat one of the sequences of evolving vectors as a gauge evolution of the "measuring device". We can reproduce the scheme of Section 9 by viewing this sequence as a sequence of reference vectors.

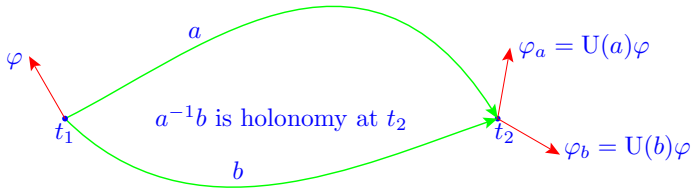

Figure 2: Different evolutions with nontrivial gauge holonomy

Figure 2 shows two different unitary evolutions of a unit vector $\varphi \in \mathcal{H}$. A necessary condition for the existence of observable physical effects is the non-triviality of holonomy around the loop formed by these evolutions.

Namely, we have

$$\langle \varphi_a \mid \varphi_b \rangle = \left\langle \varphi \left| \mathsf{U}\left(a^{-1}b\right) \right| \varphi \right\rangle , \tag{12}$$

where $a, b \in \mathsf{G}$ and U is a unitary representation of G in $\mathcal{H}$. If the holonomy $h = a^{-1}b$ is trivial (i.e. $h = \boldsymbol{e} \in \mathsf{G}$) then $\langle \varphi_a \mid \varphi_b \rangle = \langle \varphi \mid \varphi \rangle \equiv 1$.

In differential geometry, the infinitesimal version of holonomy is called the *curvature* of a connection. In the gauge theories, the fundamental physical forces are described as the curvatures of certain gauge connections.

Consider in the infinitesimal approximation the discrepancy — measured by the Born rule[4] — between the two evolutions depicted in Figure 2. In this approximation we assume that G is a continuous group and use the Lie algebra (i.e. linear) approximation. Thus, we have $\mathsf{U}(h) \approx \mathrm{I} + \mathrm{i}\, F$, where I is the unit matrix, F is a Hermitian matrix. Approximating scalar product (12) we have $\langle \varphi_a \mid \varphi_b \rangle \approx \langle \varphi \left| \mathrm{I} + \mathrm{i}\, F \right| \varphi \rangle = 1 + \mathrm{i} \langle \varphi \left| F \right| \varphi \rangle$. Here we encounter an artifact of the approximation: whereas the vector $\mathsf{U}(h)\,\varphi$ due to the unitarity has unit length, its approximation, $(\mathrm{I} + \mathrm{i}\, F)\,\varphi$, necessarily has a length greater than 1. Thus, for calculating the Born formula we need to introduce the normalizing coefficient:

$$\left\| (\mathrm{I} + \mathrm{i}\, F)\, \varphi \right\|^2 = \langle \varphi \left| (\mathrm{I} - \mathrm{i}\, F)\, (\mathrm{I} + \mathrm{i}\, F) \right| \varphi \rangle = 1 + \left\langle \varphi \left| F^2 \right| \varphi \right\rangle .$$

Now the infinitesimal discrepancy up to the quadratic terms has the form

$$\mathbb{P}_h \approx \frac{1 + \langle \varphi \left| F \right| \varphi \rangle^2}{1 + \langle \varphi \left| F^2 \right| \varphi \rangle} \approx 1 - \left\langle \varphi \left| F^2 \right| \varphi \right\rangle + \langle \varphi \left| F \right| \varphi \rangle^2 . \tag{13}$$

The approximate entropy follows from (13):

$$\mathbf{S}_h = -\ln \mathbb{P}_h \approx \left\langle \varphi \left| F^2 \right| \varphi \right\rangle - \langle \varphi \left| F \right| \varphi \rangle^2 \equiv \left(\Delta_\varphi F \right)^2 . \tag{14}$$

In probability theory and statistics, the values $\Delta_\varphi F$ and $\left(\Delta_\varphi F\right)^2$ are called the *standard deviation* and the *variance*, respectively. Thus, equation (14) reveals the relationship between the curvature of the gauge connection and the quantum uncertainty.

Energy of permutations

Planck's formula $E = h\nu$ associates the energy with the periods of underlying microscopic processes. By analogy, we define the *"energy"* of a permutation p as its frequency: $\boldsymbol{\varepsilon}_p = \dfrac{1}{\mathrm{ord} p}$. Of course, this definition is only

[4] Gleason's theorem proves that the only possible measure on the subspaces of a Hilbert space is defined by the Born rule.

approximate because the unitary operator associated with the permutation contains many frequencies: they are inverses of the lengths $\ell_1, \ldots, \ell_M$ of disjoint cycles that constitute the permutation, and the period of the permutation is equal to $\mathrm{ord} p = \mathrm{lcm}\,(\ell_1, \ldots, \ell_M)$. However, the definition is useful for classifying the elements of permutation groups from a "physical point of view".[5] Moreover, the permutations of the dominant conjugacy classes usually contain a small number (typically 1) of dominating frequencies and the definition becomes "almost exact". Figure 3 shows the time

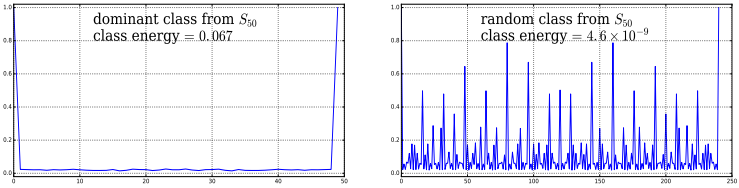

Figure 3: Zeno probability evolution for dominant and randomly chosen classes.

evolutions of the probability to observe an initial vector $\varphi_{t=0}$ ("Zeno probability") for the elements from the dominant and randomly chosen conjugacy classes of the group S_{50}. The evolutions of vectors from 49-dimensional standard representation are considered. We see that the elements from the dominant class provide quite regular evolutions with a sharp single frequency, whereas evolutions provided by the elements of the random class contain many different frequencies. On the other hand, the energy of the dominant class — we define the energy of a class C_p that contains a permutation p as $E_p = \varepsilon_p |C_p|$ — is much larger than the energy of the random class, namely, $0.067 \gg 4.6 \times 10^{-9}$.

Monte Carlo simulation

Some simple tasks, such as constructing the energy spectra, can be done by exact computing for the groups up to about S_{70} ($|\mathsf{S}_{70}| \approx 1.2 \times 10^{100}$). However, to deal with the larger groups or more complicated problems, e.g. study of quantum evolutions with taking into account the gauge

[5] The statistical meaning of energy is that the probability to observe a group element at a given time is proportional to its frequency, so energy can be regarded as a "statistical weight" of the element.

connections, more efficient tools are needed. It is natural to resort to Monte Carlo methods. To compare Monte Carlo simulation with exact computation consider the group S_{50}. Its numerical characteristics are: $|\mathsf{S}_{50}| = 50! \approx 3.04 \times 10^{64}$; number of conjugacy classes $= 204226$; number of different periods $= 1056$; maximum period $= 180180$. Figure 4

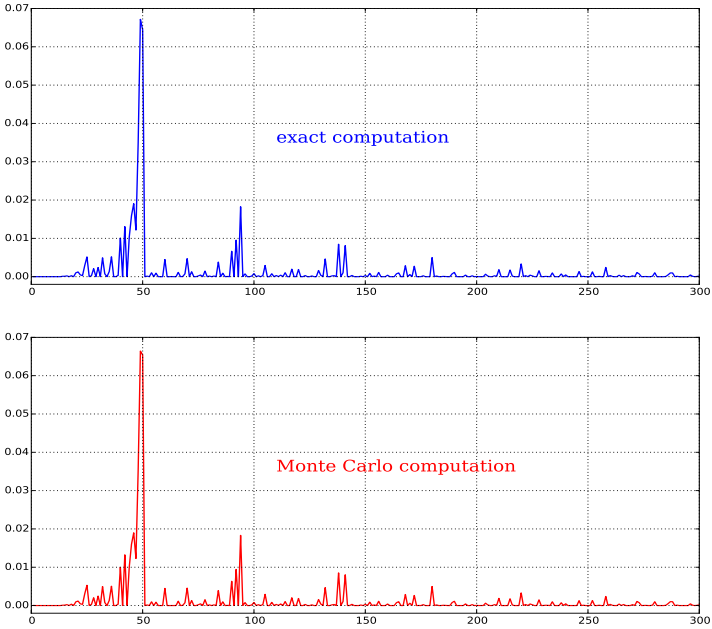

Figure 4: Energy spectrum of S_{50}. Exact vs Monte Carlo computations.

shows the "energy spectra" obtained by both exact and Monte Carlo computations for the dominant in each period classes in the range of periods [1..300]. We have generated 10^6 random elements of the group S_{50}, which is a negligible part ($\approx 3 \times 10^{-59}$) of the whole population. However, as it is seen in Figure 4, even the finest details of both spectra coincide quite perfectly. The task took about 10 sec on a 3.3 GHz PC. Thus, we see that the Monte Carlo method is highly precise and efficient in our studies.

Concluding remarks

The term "emergence" refers to the appearance of regular structures in large collections of more elementary entities or processes. The typical examples in physics are the phase transitions, crystals, convection cells etc. Some physical theories suggest that many structures, that are viewed traditionally as fundamental, e.g. *space* or *mass*, are in fact emergent phenomena and can be derived from more primitive underlying elements. Some philosophical considerations allows to suggest that any specific (i.e. non-generic) structure must appear in the nature as an emergent phenomenon. From a mathematical point of view, the emergent phenomena are the most likely configurations of the elementary constituents (or, more generally, the dominant configurations, if nontrivial weights are involved). Our main goal is to develop methods to search for specific emergent phenomena in various problems of quantum mechanics. We hope that the Monte Carlo method can find the dominant structures and evolutions efficiently.

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